Problem Set 6

Please label the problems with Chapter/Section/Problem (e.g. Exercise 2.3.42, Exercise 2.3.54, etc) in your homework.

Announcement

Test 3 is on Monday August 1, 2005

Final Exam is on Thursday August 4, 2005
4:30pm — 6:30pm, LCB 121

1. Exercise 5.1.16.
2. Exercise 5.1.18.
3. Exercise 5.1.28.
4. Exercises 5.2.14, 5.2.28
5. Exercise 5.2.30.
6. Exercise 5.4.34.
7. Exercise 5.5.10.
8. Exercise 5.5.12.
   (Here, we work over the interval $[-\pi, \pi]$ and the Fourier coefficients are those with respect to 1, $\sin(t)$, $\cos(t)$, $\sin(2t)$, $\cos(2t)$, $\ldots$, etc.)
9. Exercise 5.5.18.
10. Exercises 5.5.26, 5.5.28
11. Let V be an inner product space over $\mathbb{R}$ and U a subspace of V.
    a) Prove that U is always a subspace of $U^{\perp\perp} := (U^{\perp})^{\perp}$.
    b) Suppose $\dim(U) < \infty$. Prove that for each $\vec{x} \in V$, we have $||\text{proj}_U(\vec{x})|| \leq ||\vec{x}||$.
    c) Suppose $\dim(U) < \infty$. Prove that the map defined by
    $$\text{proj}_U : V \longrightarrow V : \ \vec{x} \longmapsto \text{proj}_U(\vec{x})$$
    is a linear map.
    d) Suppose $\dim(V) < \infty$. Prove that $\dim(V) = \dim(U) + \dim(U^{\perp\perp})$.
    e) Suppose $\dim(V) < \infty$. Prove that $U = U^{\perp\perp}$.

Please turn over.

12. **Maple Problem** Define $f(t)$ as follows:

$$f(t) := \begin{cases} -t-1, & -\pi \le t \le 0 \\ -t+1, & 0 \le t \le \pi \end{cases}$$

a) Download the two "Fourier approximation" files from the course web page. The second one is an executable XMaple worksheet while the first one is the same worksheet in `pdf` format.

b) Use Xmaple, mimicking the downloaded worksheet, compute the Fourier approximations of $f(t)$ over the interval $[-\pi, \pi]$ with respect to the L_2 inner product and each of the following spanning sets:

i) $[\ \sin(t), \sin(2t)\ ]$

ii) $[\ \sin(t), \sin(2t), \ldots, \sin(7t)\ ]$

iii) $[\ \sin(t), \sin(2t), \ldots, \sin(20t)\ ]$

iv) $[\ t, t^3, t^5, t^7\ ]$

v) $[\ t, t^3, t^5, t^7, \ldots, t^{39}\ ]$

Compute the error for each Fourier approximation. Plot each approximation and $f(t)$ on the same graph.

c) Submit your worksheet as usual with the rest of your problem set.

Remarks

- When defining the function $f(t)$ above in Maple, you may find the built-in `Maple` function `Heaviside` useful. Look it up.
- A function $g(t)$ is said to be "odd" if $g(t)$ satisfies $g(-t) = -g(t)$ for all t, and $g(t)$ is said to be "even" if $g(-t) = g(t)$ for all t. Note $f(t)$ is an odd function as well as $\sin(mt)$ and all odd powers of t, whereas $\cos(mt)$ and all even powers of t are even functions. Convince yourself that any Fourier approximation of an odd function does not contain the even terms $\cos(mt)$ or t^{2m}.
- Note that approximation gets better when more terms are included.
- Note how the "sine" approximations are "stuck" at zero at the end points of the interval $[-\pi, \pi]$, while the "polynomial" approximations do not have this shortcoming. Think about why this is so.
- We are using differentiable functions (trigonometric functions and polynomials; they are differentiable infinitely many times) to approximate the function $f(t)$, which is not even continuous. We expect the approximation to be particularly inaccurate near points of discontinuity. Notice how each Fourier approximation "overshoots" at the point of discontinuity of $f(t)$. This phenomenon is called the Gibbs phenomenon.