Problem Set 5

Please label the problems with Chapter/Section/Problem (e.g. Exercise 2.3.42, Exercise 2.3.54, etc) in your homework.

1. Exercise 7.4.10.
2. Exercise 7.4.16.
3. Exercise 7.2.22.
4. Exercise 7.2.34. Assume the 4 × 4 matrix A has real entries. You need to deal with the possibility of complex eigenvalues, namely complex roots of the characteristic polynomial. Reminder: Complex roots of a polynomial always come in conjugate pairs.
5. Exercise 7.3.20.
6. Exercise 7.3.24.
7. Exercise 7.4.22.
8. Exercise 7.4.26.
9. a) **Definition** Two square matrices $A, B \in \mathbb{F}^{n\times n}$ are said to be *similar* if there exists an invertible square matrix $P \in \mathbb{F}^{n\times n}$ such that $B = PAP^{-1}$.
   Let $A, B, C \in \mathbb{F}^{n\times n}$. Prove that if A is similar to B and B is similar to C, then A is similar to C.

   b) Exercise 7.4.36. (Hint: Check that they can be diagonalized to the same diagonal matrix, and observe that any diagonalizable matrix is similar to any of its diagonalizations.)
10. Exercise 7.4.42.
11. **Definition** The *trace* of a square matrix is, by definition, the sum of its diagonal elements.

    Definition The *trace* of a linear map $T \in \mathrm{L}(V, V)$, where V is a finite-dimensional vector space, is, by definition,

    $$\text{trace}(T) := \text{the trace of any of its matrix representatives.}$$

    Note that the right-hand-side of the definition of the trace of a linear map above, a priori, depends on the choice of a basis for the vector space V (in order to write down a matrix representative for T).
    Prove that the trace of a linear map is well-defined (i.e. independent of the choice of the basis) as follows:

    a) Prove that for any two square matrices A and B of the same dimensions, $\text{trace}(AB) = \text{trace}(BA)$. (Hint: Use summation notation. Reminder: matrix multiplication is NOT commutative in general, i.e. $AB \neq BA$ in general. This exercise shows that the traces of the two indicated products are always the same, even though the products themselves may be different.)

    b) Prove that for any square matrix A and invertible matrix P of the same dimensions, we have $\text{trace}(PAP^{-1}) = \text{trace}(A)$. (Hint: Use the result above. Don't use summation.)

    c) Prove that every two matrix representatives of the same linear map $T \in \mathrm{L}(V, V)$ have the same trace. This completes the proof of the well-definition of the trace of a linear map $T \in \mathrm{L}(V, V)$. (Hint: Remind yourself how the matrix representative of a linear map changes as you change basis. See how you can apply the results above.)

Suggested Problem

Do NOT submit.

1. **Theorem** A linear map from a finite-dimensional vector space over $\mathbb{C}$ to itself is diagonalizable if and only if the geometric multiplicity of each of its eigenvalues is equal to the corresponding algebraic multiplicity.

   We proved in class that if equality of each corresponding geometric and algebraic multiplicity implies diagonalizability. Complete the proof by proving the reverse implication. Note that this implication does not require the base field to be $\mathbb{C}$. The fact that the base field is $\mathbb{C}$ is used only in the forward implication when we invoke the Fundamental Theorem of Algebra.