Problem Set 4

Please label the problems with Chapter/Section/Problem (e.g. Exercise 2.3.42, Exercise 2.3.54, etc) in your homework.

1. Exercises 4.2.34, 4.2.58.
2. Exercise 4.2.36.
3. Exercise 4.2.70.
4. Exercise 4.2.78.
5. Exercise 4.2.80.
6. Exercises 4.3.14, 4.3.44
7. Exercise 4.3.30.
8. Exercise 4.3.38.
9. Exercise 4.3.50.
10. Exercise 4.3.58. Note: $\vec{x} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is, by definition, the real number $x_1 + x_2 - x_3$.
11. Exercises 4.3.60 (a), (b).
12. Exercise 4.3.64.
13. **Definition** An (arbitrary) map $f : X \longrightarrow Y$ from a set X to a set Y is said to be *invertible* if there exists a map $g : Y \longrightarrow X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$, where 1_X and 1_Y are the identity maps on X and Y respectively.

    a) Prove that a map is invertible if and only if it is bijective.

    b) Prove that a linear map is invertible if and only if it is an isomorphism.

    c) Suppose $f : X \longrightarrow Y$ is an invertible map. Prove that the map $g : Y \longrightarrow X$ as in the above definition is unique. This map is called the *inverse map* of f and will be denoted by f^{-1}.

    d) Prove that the inverse map of an invertible linear map is itself linear.

14. Let U, V, W be finite-dimensional vector spaces (over $\mathbb{F}$) and $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ be bases for them respectively. Let $f \in \mathrm{L}(U, V)$ and $g \in \mathrm{L}(V, W)$.

    a) Recall that $g \circ f \in \mathrm{L}(U, W)$. Express $\mathrm{M}_{\mathcal{G}}(g \circ f(\mathcal{E}))$ in terms of $\mathrm{M}_{\mathcal{F}}(f(\mathcal{E}))$ and $\mathrm{M}_{\mathcal{G}}(g(\mathcal{F}))$.

    b) Suppose $f \in L(V, W)$ is invertible. Express $\mathrm{M}_{\mathcal{F}}(f^{-1}(\mathcal{G}))$ in terms of $\mathrm{M}_{\mathcal{G}}(f(\mathcal{F}))$

Please turn over.

15. **Maple Problem** Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) := \begin{bmatrix} 3x + 2y - 4z \\ x - 5y + 3z \end{bmatrix}$$

Use **Xmaple** or **Maple** to find $\mathrm{M}_{\mathcal{C}}(T(\mathcal{B}))$, where

$$\mathcal{B} := \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} := \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$$

as follows: Let $\mathcal{E}$ and $\mathcal{F}$ be the "standard" bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively, i.e.

$$\mathcal{E} := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{F} := \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Recall that we have the general formula:

$$\mathrm{M}_{\mathcal{C}}(T(\mathcal{B})) = \mathrm{M}_{\mathcal{C}}(\mathcal{F}) \cdot \mathrm{M}_{\mathcal{F}}(T(\mathcal{E})) \cdot \mathrm{M}_{\mathcal{E}}(\mathcal{B})$$

a) Determine each of the three matrices in the right-hand-side of the above equation, and define them in **Maple** using the command `linalg[matrix]`.
(Reminder: It is easy to read off $\mathrm{M}_{\mathcal{E}}(\mathcal{B})$ and $\mathrm{M}_{\mathcal{F}}(T(\mathcal{E}))$, but it is not so easy for $\mathrm{M}_{\mathcal{C}}(\mathcal{F})$. However, recall that $\mathrm{M}_{\mathcal{C}}(\mathcal{F}) = \mathrm{M}_{\mathcal{F}}(\mathcal{C})^{-1}$ and reading off $\mathrm{M}_{\mathcal{F}}(\mathcal{C})$ is again easy. Hence, you may find the command `linalg[inverse]` useful.)

b) Compute the required matrix product using the command `evalm`. (Do NOT use `multiply`.)

c) Submit your **Maple** worksheet along with the rest of your problem set.

Remark: Look up how to use the aforementioned commands in the **Maple** help pages as necessary. For example, to call up the **Maple** help page for, say, `evalm`, you can simply type at the **Maple** command prompt the following:

```
>?  evalm
```

and then hit `ENTER`.

The following are suggested exercises (for developing computational proficiency).
Do NOT submit them. They will not be graded.
However, they do contain test and examination material.

1. Exercise 4.2.8.
2. Exercise 4.2.18.
3. Exercises 4.3.24, 4.3.46.
4. Exercise 4.2.72.
5. Exercise 4.2.74.
6. Exercise 4.3.48.
7. Exercise 4.3.52.
8. Exercise 4.3.67.
9. Let V, W be finite-dimensional vector spaces and $\mathcal{E}$, $\mathcal{F}$ be bases for them respectively. Let $f \in \mathrm{L}(V, W)$. Prove that $\mathrm{rank}(f) = \mathrm{rank}(\mathrm{M}_{\mathcal{F}}(f(\mathcal{E}))$.
10. Let V, W be finite-dimensional vector spaces with $\dim(V) = \dim(W) =: n$. Let $v_1, \ldots, v_n$ form a basis for V and $w_1, \ldots, w_n$ form a basis for W. Define a map $f : V \longrightarrow W$ by:

$$f(\alpha_1 v_1 + \cdots + \alpha_n v_n) := \alpha_1 w_1 + \cdots + \alpha_n w_n, \quad \text{for all scalars } \alpha_1, \ldots, \alpha_n.$$

Prove that f is an isomorphism.