# Fields, Vector Spaces over a Field, Algebras over a Field

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| | A *field* is a set $\mathbb{F}$ equipped with<br><br>"addition" and "multiplication"<br><br>such that for all $\alpha, \beta, \gamma \in \mathbb{F}$<br><br>the following hold | A *vector space over a field* $\mathbb{F}$ is a set $V$ equipped with<br><br>"scalar multiplication", "addition" and "multiplication"<br><br>such that for all $\alpha, \beta \in \mathbb{F}$, and all $u, v, w \in V$ the following hold | An *algebra over a field* $\mathbb{F}$ is a set $\mathcal{A}$ equipped with<br><br>"scalar multiplication", "addition" and "multiplication"<br><br>such that for all $\alpha, \beta \in \mathbb{F}$, and all $A, B, C \in \mathcal{A}$ the following hold |
| Addition<br>$+$ | Commutative: $\alpha + \beta = \beta + \alpha$<br>Associative: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$<br>Zero: $\alpha + 0 = \alpha$<br>Additive inverse: $\alpha + (-\alpha) = 0$ | Commutative: $u + v = v + u$<br>Associative: $(u + v) + w = u + (v + w)$<br>Zero: $u + 0 = u$<br>Additive inverse: $u + (-u) = 0$ | Commutative: $A + B = B + A$<br>Associative: $(A + B) + C = A + (B + C)$<br>Zero: $A + 0 = A$<br>Additive inverse: $A + (-1 \cdot A) = 0$ |
| Scalar Multiplication<br>$\cdot$ | Same as multiplication; see below. | "Commutative": $\alpha \cdot u = u \cdot \alpha$<br>"Associative": $\alpha \cdot (\beta \cdot u) = (\alpha\beta) \cdot u$ | "Commutative": $\alpha \cdot A = A \cdot \alpha$<br>"Associative": $\alpha \cdot (\beta \cdot A) = (\alpha\beta) \cdot A$ |
| Multiplication<br><br>$\cdot$ | Commutative: $\alpha\beta = \beta\alpha$<br><br>Associative: $(\alpha\beta)\gamma = \alpha(\beta\gamma)$<br><br>Identity: $\alpha \cdot 1 = \alpha = 1 \cdot \alpha$<br><br>Multiplicative inverse: $\alpha \cdot (\frac{1}{\alpha}) = 1 = (\frac{1}{\alpha}) \cdot \alpha$, for $\alpha \neq 0$ | No multiplication | NOT Commutative: $A \cdot B \neq B \cdot A$, in general<br><br>Associative: $(A \cdot B) \cdot C = A \cdot (B \cdot C)$<br><br>Identity: $A \cdot 1_{\mathcal{A}} = A = 1_{\mathcal{A}} \cdot A$ |
| Distribution | $\alpha\,(\beta + \gamma) = \alpha\beta + \alpha\gamma = (\beta + \gamma)\,\alpha$ | $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u = u \cdot (\alpha + \beta)$<br>$\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v = (u + v) \cdot \alpha$ | $(\alpha + \beta) \cdot A = \alpha \cdot A + \beta \cdot A = A \cdot (\alpha + \beta)$<br>$\alpha \cdot (A + B) = \alpha \cdot A + \alpha \cdot B = (A + B) \cdot \alpha$<br><br>$A \cdot (B + C) = A \cdot B + A \cdot C$<br>$(B + C) \cdot A = B \cdot A + C \cdot A$ |