Row-reduced Echelon Form of a Matrix

Example Solve

$$\begin{cases} x + y + z - w = 0 \\ 2x + 2y + z - 3w = 0 \\ -x - y + z + 3w = 0 \end{cases}$$

$$A := \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 2 & 1 & -3 \\ -1 & -1 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - 2R_1 \\ R_3 + R_1 \end{matrix}$$

$$A_0 := \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ -R_2 \\ R_3 + 2R_2 \end{matrix}$$

So, the given system is equivalent to

$$\begin{cases} x + y - 2w = 0 \\ z + w = 0 \\ 0 = 0 \end{cases},$$

whose solution is:

$$(x, y, z, w) = (-s - 2t,\ s,\ t,\ -t), \quad s, t \in \mathbb{R}$$

Remarks

- The matrix A is not in rref, while A_0 is.
- It should be clear by now that: Every matrix M can be transformed to a UNIQUE matrix in row-reduced echelon form via row operations. This matrix is denoted by rref(M).
- Note that rref(M) should be regarded as some kind of a "standard simplification" of M.
- **Definition** Let M be a matrix. Then

  $$\text{rank}(M) := \text{the number of nonzero rows in rref}(M)$$

- Notice the interpretation: rank(M) = the number of "independent" rows of the matrix M.
- Suppose $M \in \mathbb{C}^{m \times n}$. Then M can be regarded as the coefficient matrix of a system of m linear equations in n unknowns.

  The "effective" number of equations of the that system is precisely rank(M). rank(M) is the number of non-redundant equations in that system.

  The number of parameters of the solution(s) of that system is precisely $n - \text{rank}(M)$, namely, the number of unknowns minus the "effective" number of equations.