Inner Products and Their Associated Norms

Definition Let $(V, \langle\, ,\, \rangle)$ be an inner product space over $\mathbb{R}$.

1. The *norm* associated to $\langle\, ,\, \rangle$ is, by definition,

$$||\cdot|| : V \longrightarrow [0, \infty), \quad ||\vec{v}|| := \sqrt{\langle\, \vec{v},\, \vec{v}\rangle}, \quad \text{for } \vec{v} \in V.$$

2. The *distance* between $\vec{u}, \vec{v} \in V$ is $||\vec{u} - \vec{v}||$.

Observations

1. $||\vec{v}|| = 0 \Longleftrightarrow \vec{v} = \vec{0}$.
2. $||\vec{u} - \vec{v}|| = 0 \Longleftrightarrow \vec{u} = \vec{v}$.
3. $||\alpha\vec{v}|| = |\alpha|\, ||\vec{v}||$ for all $\alpha \in \mathbb{R}$ and for all $\vec{v} \in V$.
   Indeed, $||\alpha\vec{v}||^2 = \langle\, \alpha\vec{v}, \alpha\vec{v}\,\rangle = \alpha^2\langle\, \vec{v}, \vec{v}\,\rangle = \alpha^2\, ||\vec{v}||^2 \Longrightarrow ||\alpha\vec{v}|| = |\alpha|\, ||\vec{v}||$.

Cauchy-Schwarz Inequality
Let $(V, \langle\, ,\, \rangle)$ be an inner product space over $\mathbb{R}$, with associated norm $||\cdot||$. Then,

$$|\langle\, \vec{u}, \vec{v}\,\rangle| \le ||\vec{u}||\, ||\vec{v}||, \quad \text{for all } \vec{u}, \vec{v} \in V.$$

Proof Adopt the proof from the case of the Euclidean dot product. $\square$

Recall that the Cauchy-Schwarz Inequality allowed us to define the (cosine of the) angle between two non-zero vectors in Euclidean space:

$$\cos\angle(\vec{x}, \vec{y}) := \frac{\vec{x} \bullet \vec{y}}{||\vec{x}||\, ||\vec{y}||} \in [-1, 1].$$

Since we now know the Cauchy-Schwarz Inequality holds in a general inner product space, we can define *angles* and *orthogonality* the same way we did for Euclidean spaces:

For non-zero $\vec{u}, \vec{v} \in V$,

$$\cos\angle(\vec{u}, \vec{v}) := \frac{\langle\, \vec{u}, \vec{v}\,\rangle}{||\vec{u}||\, ||\vec{v}||} \in [-1, 1].$$

And, $\angle(\vec{u}, \vec{v})$ is the unique angle in $[0, \pi]$ such that

$$\cos\left(\angle(\vec{u}, \vec{v})\right) = \frac{\langle\, \vec{u}, \vec{v}\,\rangle}{||\vec{u}||\, ||\vec{v}||}.$$

$\vec{u}, \vec{v} \in V\ \{\vec{0}\}$ are said to be *orthogonal* if $\langle\, \vec{u}, \vec{v}\,\rangle = 0$.