Problem Set 0

1. Solve $|x+7| = 5$.
2. Given $x + \frac{1}{x} = a$, find $x^2 + \frac{1}{x^2}$.
3. Simplify $\frac{(x^{m-n})^{m+n-1}}{(x^m)^m} \cdot x^{m^2+n^2}$.
4. Simplify $\frac{1}{1-x^{a-b}} - \frac{1}{x^{b-a}-1}$.
5. Show that $\sqrt{7} + \frac{9}{4+\sqrt{7}}$ is *not* an irrational number.
6. Prove the **Remainder Theorem**:

   When a polynomial $p(x)$ is divided by a (linear) polynomial $ax - b$, the remainder is $p(\frac{b}{a})$, i.e. the polynomial p evaluated at $x = \frac{b}{a}$.

   a) Observe that using long division of polynomials, we can find polynomials $Q(x)$ and $R(x)$ such that
   $$p(x) \equiv (ax-b)Q(x) + R(x). \tag{1}$$
   What is the degree of $R(x)$ (when the long division is complete)?

   b) Determine $R(x)$ by evaluating (1) at an appropriate value of x.

7. The following result is called the **Factor Theorem**:

   a) Prove that: *If $ax - b$ is a factor of the polynomial $p(x)$, then $p(\frac{b}{a}) = 0$.*
   Hint: If $ax - b$ is a factor of $p(x)$, then we can write $p(x)$ as a product of ...?

   b) Prove that: *If $p(\frac{b}{a}) = 0$, then $ax - b$ is a factor of $p(x)$.*
   Hint: Use the Remainder Theorem.

8. Factor $p(x) := 4x^3 - 4x^2 - 9x + 9$ completely as follows:

   a) Explain why it is helpful to look for factors of $p(x)$ of the form $ax - b$ where a and b are integers, a divides 4 and b divides 9. (Note that both a and b could be negative.)
   Hint: Examine the leading coefficient and the constant term of $p(x)$.

   b) So, the only possibilities are $a = \pm 1, \pm 2$ and $b = \pm 1, \pm 3$. (Why?) Now, use the Factor Theorem to determine which combinations indeed lead to factors of $p(x)$.

   c) Completely factor $p(x)$.