

# **A Special Hour with Relativity**

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# Abstract

- What prompted Einstein: Incompatibilities between Newtonian Mechanics and Maxwell's Electromagnetism.
- We will state the spacetime hypotheses and the postulates Special Relativity and use them to derive the transformation law between inertial frames in Special Relativity, which implies that the Minkowski metric on spacetime can be defined.
- We will discuss four phenomenon predicted by Special Relativity: time dilation, length contraction, breakdown of universal simultaneity and the twin paradox.

## Newtonian mechanics is relativistic too!

Newton's Second Law:  $F = m\ddot{x}$

Galilean Transformation between *inertial* observers in Newtonian Mechanics:

$$\begin{cases} t' = t \\ x' = x - vt, \end{cases}$$

where  $v$  is the constant relative speed between the two observers.

Since the Second Law involves only  $\ddot{x}$ , it holds for both observers.

Roughly speaking, inertial observers are *non-accelerating* observers. Their observations are “not contaminated” by acceleration of their own reference frames.

## Relativity is a Good Thing.

The invariance of Newton's Second Law under Galilean Transformations is called Newtonian Relativity.

Newton, Einstein, and basically physicists of all time consider that natural laws should take the same form for any *inertial* observers — i.e. all physical laws should exhibit relativity.

Mathematically, *relativity* means that the equations describing any physical law should be invariant under the group of transformations between *inertial* observers.

## **The crux is: How do inertial frames transform?**

The truth is: Newton didn't know!

He adopted Galilean transformations basically because he wanted (Newtonian) relativity and he realized that his Second Law is invariant under Galilean transformations.

And, of course, Galilean transformations were the absolutely reasonable thing to adopt based on “everyday” experience — all that was available to Newton at the time.

## Newton's "mistake": He assumed universal time!

Recall Galilean Transformation: 
$$\begin{cases} t' = t \\ x' = x - vt, \end{cases}$$

The adoption of the first equation in effect tacitly (and wrong in light of Special Relativity) assumes the existence of *universal time*:

It is possible to define a *universal time* agreed by all inertial observers in the entire universe as follows: Equip all inertial observers with identical and synchronized clocks. Then these clocks stay synchronized everywhere in the universe, for all time, for all inertial observers!

Phrased as such, it should be obvious that  $t' = t$  is in fact a huge assumption — and Newton knew it!

## Signs of Weakness of Newtonian Mechanics

At the turn of the last century, there arose a new rising star among the physical theories:

Maxwell's Electromagnetism

Lorentz derived the transformations under which Maxwell's equations are invariant. They are called the Lorentz transformations.

The VERY Bad News Is:

Lorentz transformations are different from Galilean transformations!

## Signs of Weakness of Newtonian Mechanics (cont'd)

Also, Maxwell's equations predict that the speed of light in a vacuum is a constant ( $\approx 2.99 \times 10^8$  m/s) — for EVERY observer for whom Maxwell's equations hold!

But Newtonian mechanics predicts that the speed of a light signal (or any moving object) as observed by two inertial observers must differ by the relative speed of the two observers.

In a nutshell, Newtonian Mechanics and Maxwell's Electromagnetism are incompatible — unless relativity is to fail for one of them.

## Three Obvious Options

1. There is an *ether*, the medium in which light travels, which provides a preferred inertial frame for electromagnetism. Both Newton and Maxwell were right, but relativity fails for electromagnetism. Maxwell's equations hold only in the Ether frame.
2. Maxwell was wrong. Relativity holds for both Mechanics and Electromagnetism, but Maxwell's electromagnetism needs to be modified.
3. Newton was wrong. Relativity holds for both Mechanics and Electromagnetism, but Newton's mechanics needs to be modified.

## **The Michelson-Morley and Similar Experiments**

Assume the Ether exists.

Newtonian relativity implies that inertial observers travelling at a non-zero speed with respect to the Ether frame must measure a different speed of light.

A vast amount of effort was spent to devise ever more sophisticated experiments to measure these discrepancies of light speed.

But, every single one failed: The measured light speed was invariably that same constant predicted by Maxwell!

**The Emission Theories:  
Attempts to modify Maxwell's Electromagnetism**

It suffices to say: All these attempts were contradicted by experiments.

**In comes Einstein!**

## Special Relativity “=” Minkowski Geometry

The mathematics of Special Relativity is Minkowski geometry:

(Gravity-free) spacetime is modelled by  $\mathbb{R}^4$  equipped with the following distance function (metric):

Let  $P_1$  and  $P_2$  be two points in spacetime (will be called *events*).

Let  $P_1 = (t_1, x_1, y_1, z_1)$  and  $P_2 = (t_2, x_2, y_2, z_2)$  be the respective coordinates of these two points as measured by any fixed inertial observer.

Then the (possibly negative!!) “distance” between  $P_1$  and  $P_2$  is given by:

$$|P_1 - P_2|^2 = (t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2.$$

## Special Relativity “=” Minkowski Geometry (cont'd)

The definition

$$|P_1 - P_2|^2 = (t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2,$$

of course, tacitly assumes that the RHS is independent of the inertial observer whose coordinates for  $P_1$  and  $P_2$  are being used.

## Our Plan: To prove the well-definition of the Minkowski metric

1. State the definition of *inertial observers*.
2. State the assumptions on Spacetime.
3. State the two postulates of Special Relativity.
4. *Derive*, using the above assumptions, the transformation law between inertial observers.

This will turn out to be the Lorentz transformations.

The Lorentz transformation law in turn implies the well-definition of the Minkowski metric on Spacetime.

## Definition of an Inertial Observer

Inertial Observer :=  $\left\{ \begin{array}{l} \text{Un-accelarated — self-test} \\ \text{Newton's First Law holds for them} \\ \text{Able to measure time intervals} \\ \text{and spatial distances} \end{array} \right.$

## Assumptions on Spacetime

1. Homogeneity: No experiment performed by any inertial observer can determine a preferred event (point) in Spacetime.
2. Isotropy: No experiment performed by any inertial observer can determine a preferred direction in Spacetime.

## The Two Postulates of Special Relativity

1. Principle of Relativity: All physical laws (not just mechanical ones, as within the framework of Newtonian mechanics) are identical in all inertial frames.

i.e. the outcome of any physical experiment is the same when performed with *identical*\* initial conditions relative to any inertial frame.

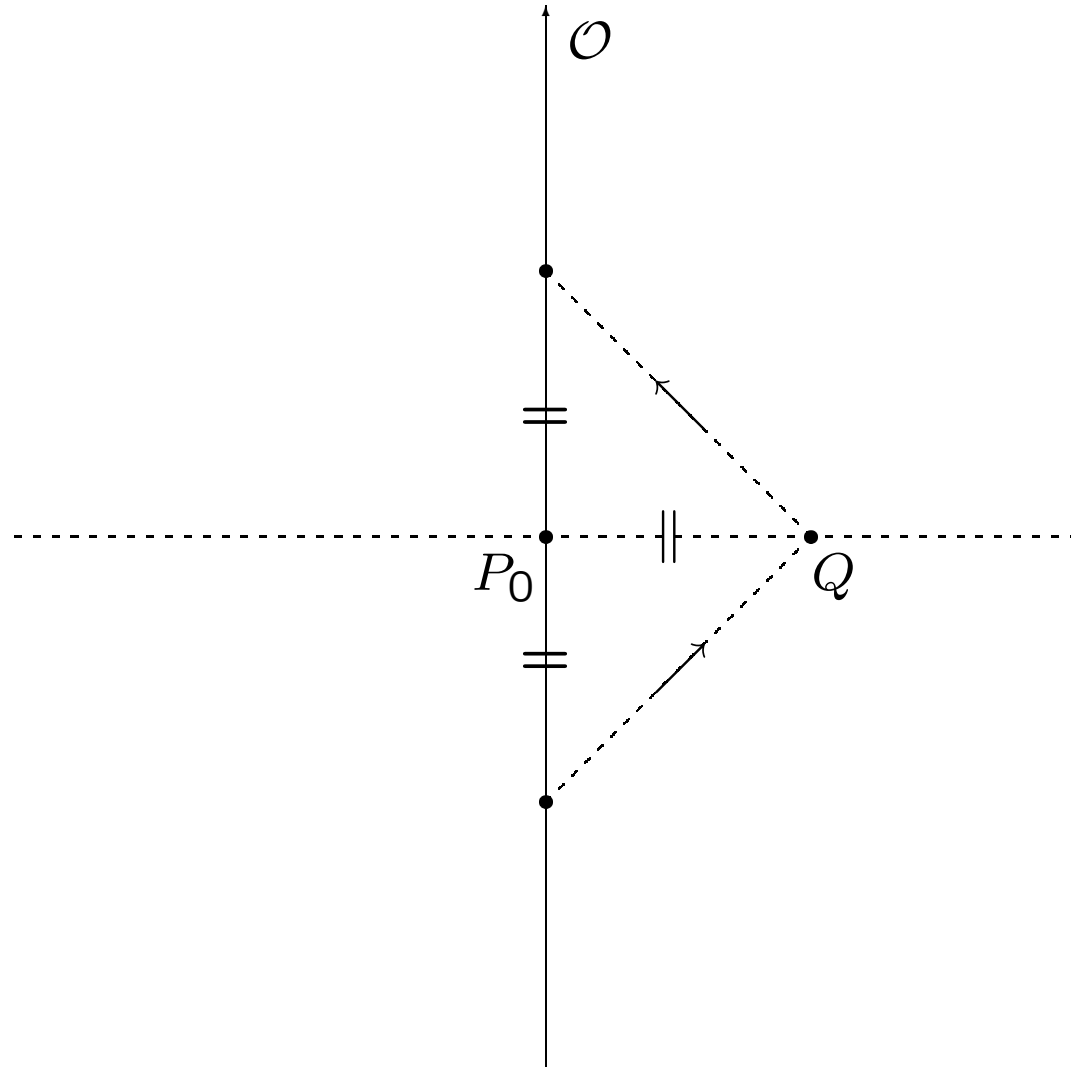
2. Constancy of the Speed of Light: Light signals in vacuum propagate rectilinearly with the same speed, at all times, in all directions, in all inertial frames.

\*with respect to the assumptions on Spacetime

## Coordinatization of Spacetime by an Inertial Observer

1. She picks an arbitrary spacetime point as origin.
2. Her ability to measure time intervals allows her to set up a time axis (her world line) in spacetime.
3. She defines simultaneity *in her own reference frame* — using the constancy of the speed of light.

# Coordinatization (cont'd)



Let  $\mathcal{O}$  and  $\mathcal{O}'$  be two inertial observers, travelling at fixed relative speed  $v$ , who coincide in spacetime at one (and only one) spacetime point.

For simplicity, we work in only one spatial dimension.

By spacetime homogeneity, we may assume the two observers take their event of coincidence as their common origin.

By spacetime isotropy, we may assume, by re-aligning the spatial axes of the two observers if necessarily, that  $\mathcal{O}'$  travels in the positive  $x$ -direction of  $\mathcal{O}$  and the two positive  $x$ -axes are aligned.

Let  $P$  be an arbitrary spacetime point and let  $P = \begin{pmatrix} t \\ x \end{pmatrix}$  and  $P = \begin{pmatrix} t' \\ x' \end{pmatrix}$  be the coordinates of  $P$  according to  $\mathcal{O}$  and  $\mathcal{O}'$  respectively.

We want to derive how the two sets of coordinates are related. Write

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = f \begin{pmatrix} t \\ x \end{pmatrix}.$$

**Claim:** Spacetime homogeneity implies that  $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$  must be linear.

Since  $\mathcal{O}$  and  $\mathcal{O}'$  have a common spacetime origin, we must have

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} at + bx \\ ct + dx \end{pmatrix}.$$

Then,

$$\begin{aligned} (t')^2 - (x')^2 &= \dots = (a^2 - c^2)t^2 + 2(ab - cd)tx + (b^2 - d^2)x^2 \\ &\stackrel{?}{=} t^2 - x^2. \end{aligned}$$

Recall

$$(t')^2 - (x')^2 = \dots = (a^2 - c^2)t^2 + 2(ab - cd)tx + (b^2 - d^2)x^2.$$

Now,

$$\text{Constancy of Speed of Light} \iff \left\{ \begin{array}{l} (t')^2 - (x')^2 = 0 \\ \updownarrow \\ t^2 - x^2 = 0 \end{array} \right\}$$

So for  $P = \begin{pmatrix} t' \\ x' \end{pmatrix} \neq \begin{pmatrix} 0' \\ 0' \end{pmatrix}$  with  $(t')^2 - (x')^2 = 0$ , we have

$$0 = [(a^2 + b^2 - c^2 - d^2) \pm 2(ab - cd)] t^2,$$

or equivalently,

$$\left\{ \begin{array}{l} a^2 + b^2 - c^2 - d^2 = 0 \\ ab - cd = 0 \end{array} \right. \quad (1)$$

The worldline of  $\mathcal{O}'$  is described by

$$x' = 0 \quad \Longleftrightarrow \quad x = vt$$

according to  $\mathcal{O}'$  and  $\mathcal{O}$  respectively.

By spacetime isotropy, the worldline of  $\mathcal{O}$  must then be described by

$$x = 0 \quad \Longleftrightarrow \quad x' = -vt'.$$

Note that minus sign!

Hence, along the worldline of  $\mathcal{O}$ :  $x = 0 \iff x' = -vt'$ ,

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} at + bx \\ ct + dx \end{pmatrix} \quad \text{becomes} \quad \begin{pmatrix} t' \\ -vt' \end{pmatrix} = \begin{pmatrix} at \\ ct \end{pmatrix},$$

which immediately implies

$$c = -av.$$

A similar consideration of the worldline of  $\mathcal{O}'$ :

$x' = 0 \iff x = vt$  implies

$$c = -dv.$$

Thus

$$\begin{aligned} -av = c = -dv &\implies a = d, \quad \text{and} \\ ab - cd = 0 &\implies b - c = 0 \implies b = -av. \end{aligned}$$

We now have

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = a(v) \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}.$$

Solving for  $\begin{pmatrix} t \\ x \end{pmatrix}$ :

$$\begin{pmatrix} t \\ x \end{pmatrix} = \frac{1}{a(v)(1-v^2)} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}.$$

But, repeating the preceding argument with the roles of  $\mathcal{O}$  and  $\mathcal{O}'$  interchanged, i.e.  $v \longleftrightarrow -v$ , we get

$$\begin{pmatrix} t \\ x \end{pmatrix} = a(-v) \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}.$$

It follows that

$$\frac{1}{a(v)(1-v^2)} = a(-v) \implies a(v) a(-v) = \frac{1}{1-v^2}.$$

**Claim:**  $a(v) = a(-v)$ .

**Q:** What is  $a(-v)$ ? **A:** It is the *time dilation factor* of  $\mathcal{O}'$ 's clock as observed by  $\mathcal{O}$ .

Indeed, the worldline of  $\mathcal{O}'$  is given by  $x = vt$  and  $x' = 0$  according to  $\mathcal{O}$  and  $\mathcal{O}'$  respectively.

$$\begin{pmatrix} t \\ vt \end{pmatrix} = a(-v) \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} t' \\ 0 \end{pmatrix} \implies$$

$$t = a(-v) t' \iff t' = \frac{t}{a(-v)}, \quad (\text{along the worldline of } \mathcal{O}')$$

which says precisely that according to  $\mathcal{O}$ , the time of  $\mathcal{O}'$  flows at the constant rate of  $1/a(-v)$  times that of  $\mathcal{O}$ . This is time dilation!

Spacetime isotropy now implies  $a(v) = a(-v)$ . **Claim 2** is proved.

So,

$$\left. \begin{aligned} a(v) a(-v) &= \frac{1}{1 - v^2} \\ a(v) &= a(-v) \end{aligned} \right\} \implies a(v) = a(-v) = \frac{1}{\sqrt{1 - v^2}}.$$

And, we finally have the transformation law we are after:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1 - v^2}} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix},$$

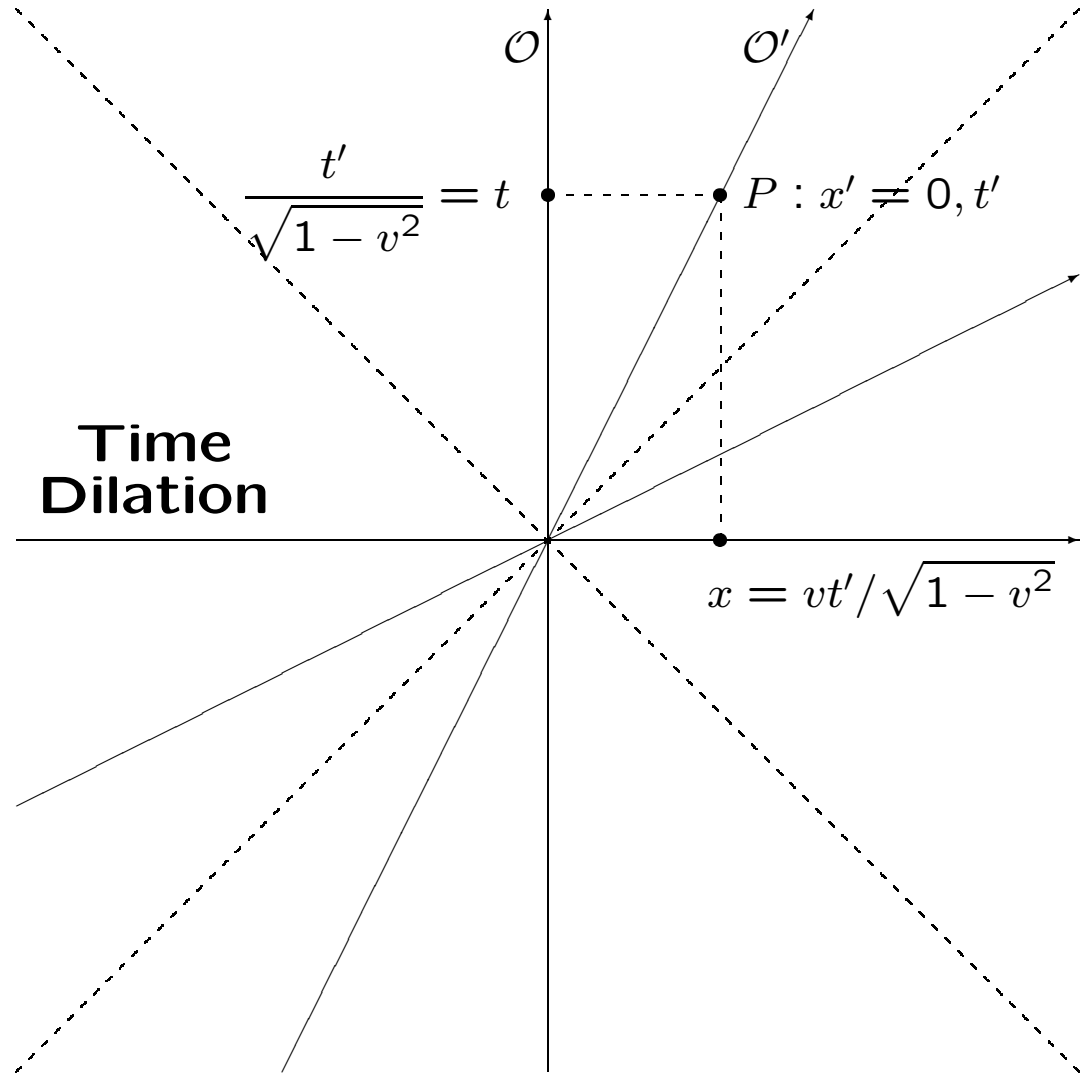
which is precisely the Lorentz transformation, under which Maxwell's equations are invariant!

**The Minkowski metric on spacetime is indeed well-defined!**

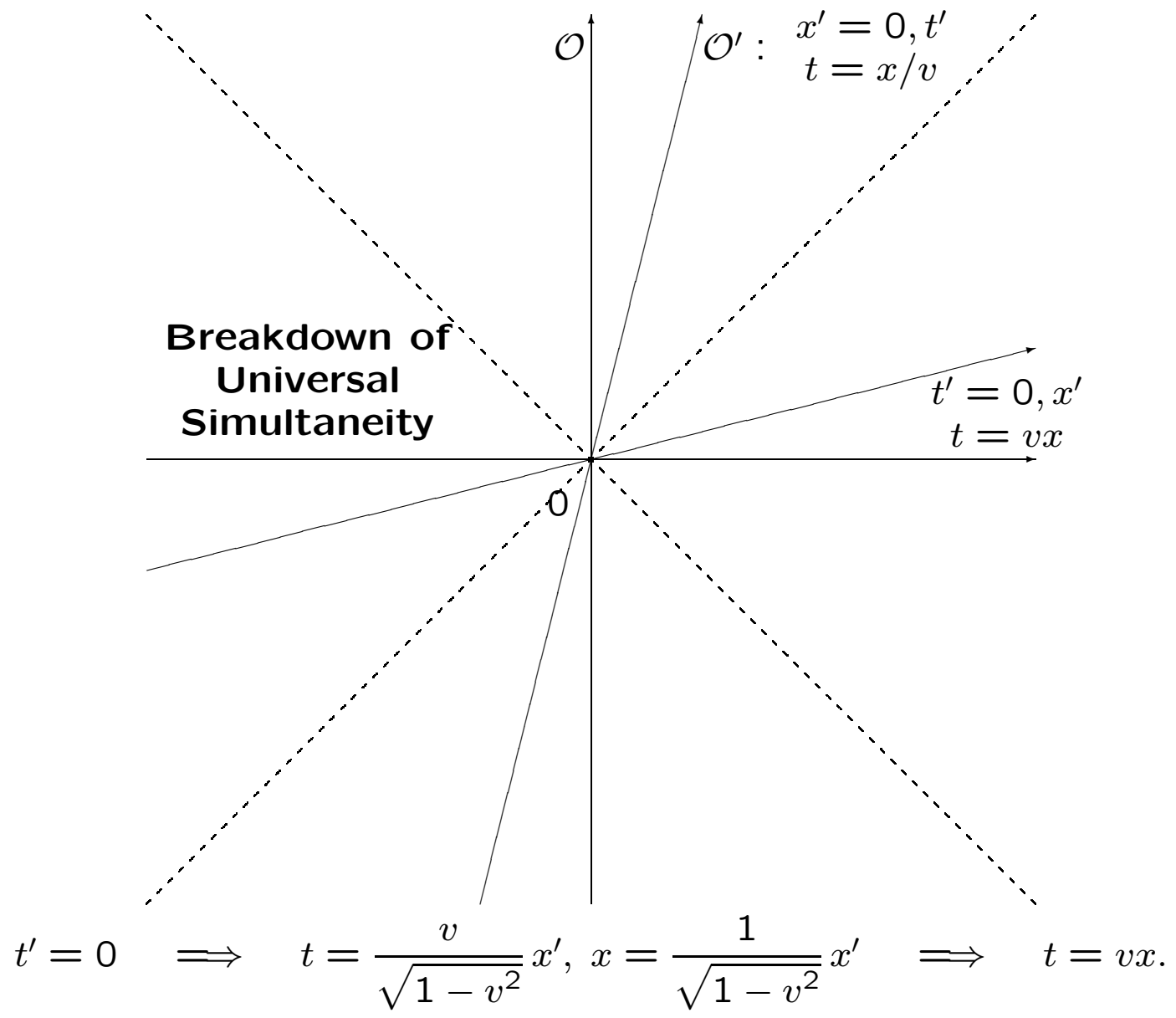
A straightforward calculation now shows that the equality

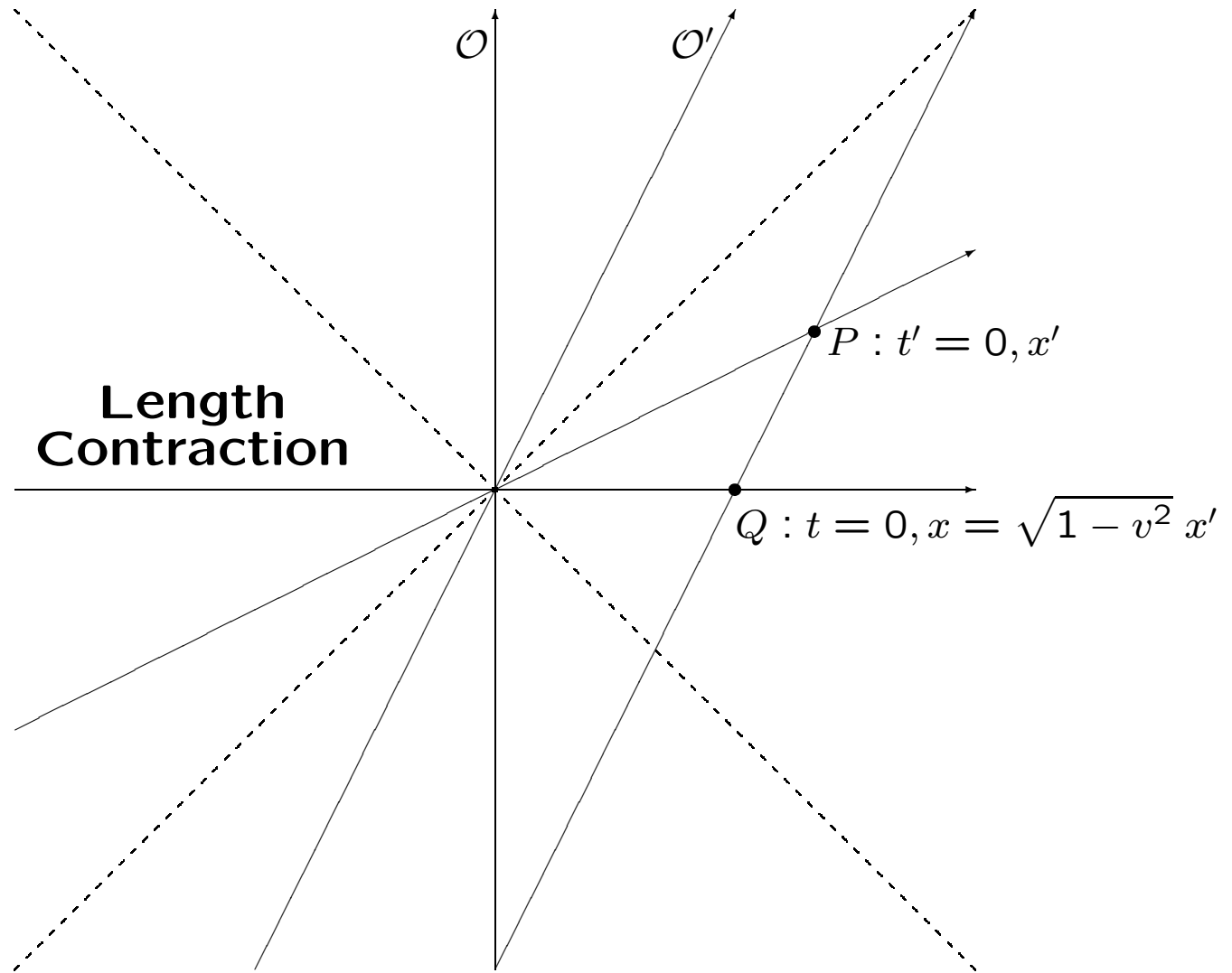
$$\begin{aligned}(t')^2 - (x')^2 &= \left( \frac{t - vx}{\sqrt{1 - v^2}} \right)^2 - \left( \frac{-vt + x}{\sqrt{1 - v^2}} \right)^2 = \dots \\ &= t^2 - x^2\end{aligned}$$

always holds if the transformation is Lorentz.



$$0 < |t'| = \sqrt{1-v^2} |t| < |t|,$$





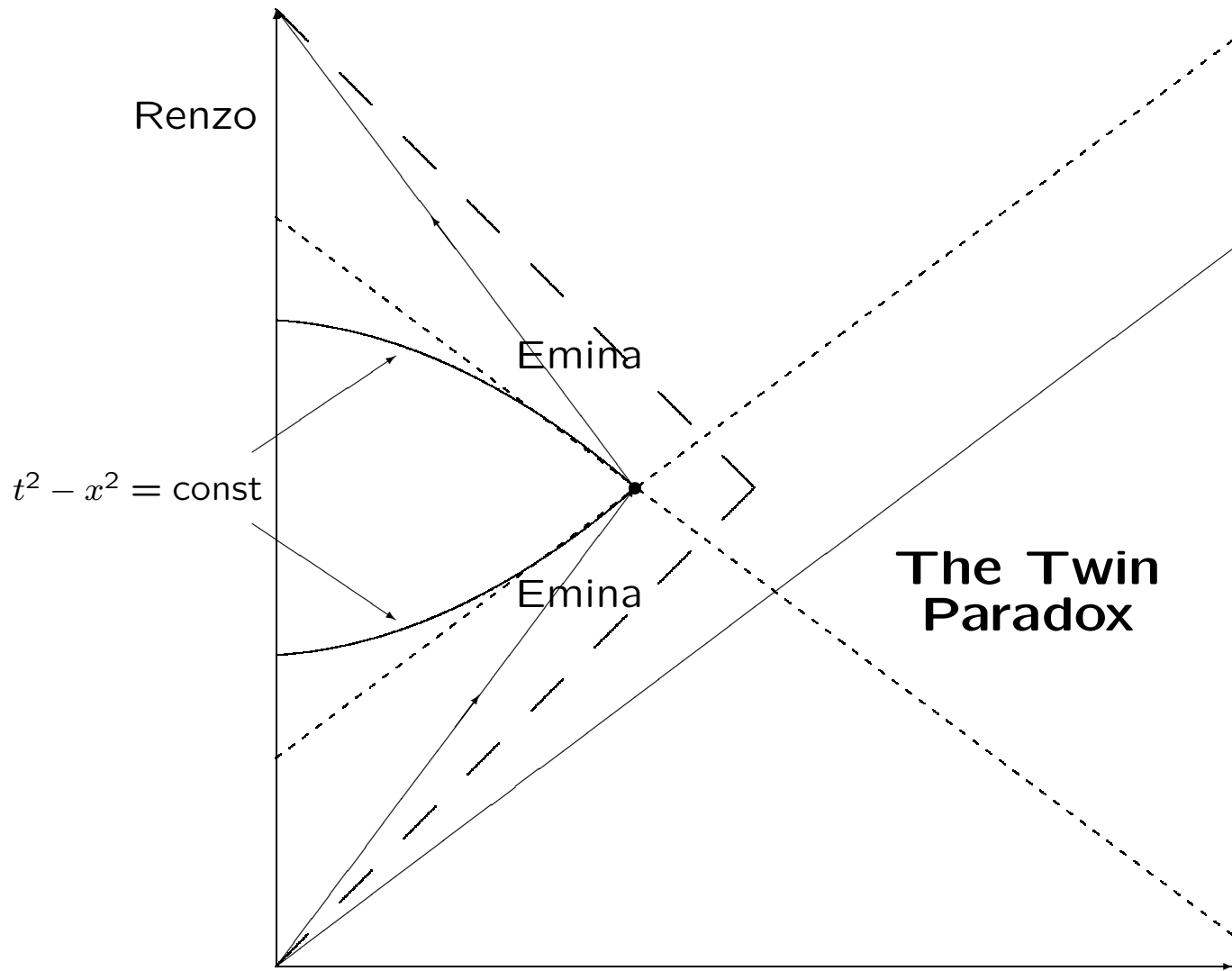
**Length  
Contraction**

$P : t' = 0, x'$

$Q : t = 0, x = \sqrt{1 - v^2} x'$

$$|OQ|_O = \sqrt{1 - v^2} |x'|$$

$$< |x'| = |OP|_{O'}$$



**\*\*\*\*\* The End \*\*\*\*\***