The Moduli Space of Real Binary Octics

Kenneth Chu

Doctoral Thesis Defense

Department of Mathematics
University of Utah
chu@math.utah.edu

March 9, 2006

The Moduli Space of Real Binary Octics

$$\begin{aligned}
\text{Binary octics} \quad &:= \quad \text{Hypersurfaces in } \mathbb{CP}^1 \text{ of degree } 8 \\
&= \quad \mathbb{P}\left(\left\{\begin{array}{c}\text{coeff. space of}\\ \text{binary octic forms}\end{array}\right\}\right) \;=\; \mathbb{P}(\mathcal{P}) \;\cong\; \mathbb{CP}^8 \\
&= \quad \left\{\begin{array}{c}\text{unordered 8-point configurations}\\ \text{in } \mathbb{CP}^1\text{, counting multiplicity}\end{array}\right\}
\end{aligned}$$

$$\text{Real} \quad := \quad \text{defining polynomial has coeff's in } \mathbb{R}$$

The Moduli Space of Real Binary Octics

$$\mathcal{M}_s^{\mathbb{R}} \quad := \quad \mathbb{P}(\mathcal{P}_s^{\mathbb{R}})/\mathbb{P}\mathsf{GL}(2,\mathbb{R})$$

$$\bigcup$$

$$\mathcal{M}_0^{\mathbb{R}} \quad = \quad \mathcal{M}_0^{\mathbb{R},0} \bigsqcup \mathcal{M}_0^{\mathbb{R},1} \bigsqcup \mathcal{M}_0^{\mathbb{R},2} \bigsqcup \mathcal{M}_0^{\mathbb{R},3} \bigsqcup \mathcal{M}_0^{\mathbb{R},4}$$

i	0	1	2	3	4
# complex conjugate pairs	0	1	2	3	4
# real points	8	6	4	2	0

Main Results

$$\mathcal{M}_0^{\mathbb{R}} \;=\; \mathcal{M}_0^{\mathbb{R},0} \bigsqcup \mathcal{M}_0^{\mathbb{R},1} \bigsqcup \mathcal{M}_0^{\mathbb{R},2} \bigsqcup \mathcal{M}_0^{\mathbb{R},3} \bigsqcup \mathcal{M}_0^{\mathbb{R},4}$$

1. $\mathcal{M}_0^{\mathbb{R},i} \;\cong\; \mathbb{P}\Gamma_i^{\mathbb{R}} \backslash \left(\mathbb{R}\mathbb{H}^5 - \mathcal{H}\right)$, $i = 0, \ldots, 4$, with $\mathbb{P}\Gamma_i^{\mathbb{R}}$ explicitly given. They are arithmetic groups.
2. $\mathbb{P}\Gamma_0^{\mathbb{R}}$, $\mathbb{P}\Gamma_1^{\mathbb{R}}$, $\mathbb{P}\Gamma_2^{\mathbb{R}}$, $\mathbb{P}\Gamma_4^{\mathbb{R}}$ are finite-index subgroups of discrete reflection groups in $\text{Isom}(\mathbb{R}\mathbb{H}^5)$.
3. The Allcock-Carlson-Toledo construction of $\mathcal{M}_s^{\mathbb{R}}$ is not a hyperbolic orbifold. (In contrast with the cases of real cubic surfaces and real binary sextics.)

The Deligne-Mostow Construction of $\mathcal{M}_s^{\mathbb{C}}$

$$\begin{array}{ccc} \mathcal{M}_s^{\mathbb{C}} & \xrightarrow{\mathfrak{p}} & \mathbb{P}\Gamma\backslash\mathbb{CH}^5 \\ \cup & & \cup \\ \mathcal{M}_0^{\mathbb{C}} & \xrightarrow{\mathfrak{p}} & \mathbb{P}\Gamma\backslash\left(\mathbb{CH}^5 - \mathcal{H}\right) \end{array}$$

$$\begin{aligned} \mathbb{P}\Gamma &:= \mathbb{P}\text{Isom}(\Lambda) \\ \Lambda &:= \left(\mathbb{Z}[\,\mathbf{i}\,]^6\,,\ \begin{bmatrix} -2 & 1+\mathbf{i} \\ 1-\mathbf{i} & -2 \end{bmatrix} \oplus \begin{bmatrix} -2 & 1+\mathbf{i} \\ 1-\mathbf{i} & -2 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1+\mathbf{i} \\ 1-\mathbf{i} & 0 \end{bmatrix} \right) \\ & \quad \text{signature}(\Lambda) \ = \ (1+, 5-) \\ \mathbb{CH}^5 &:= \mathbb{CH}\left(\,\Lambda \otimes_{\mathbb{Z}[\,\mathbf{i}\,]} \mathbb{C}\,\right) \\ \mathcal{H} &:= \bigcup \left\{ \mathbb{CH}(r^{\perp}) \subset \mathbb{CH}^5 \;\middle|\; r \in \Lambda,\ \langle r, r\rangle = -2 \right\} \end{aligned}$$

The Allcock-Carlson-Toledo Construction of $\mathcal{M}_s^{\mathbb{R}}$

$$\left\{ \begin{array}{c} \text{periods of} \\ \text{real octics} \end{array} \right\} \quad \subset \quad \bigcup_{[\chi]\in \mathbb{P}\text{IAI}(\Lambda)} \mathbb{RH}\,(\ \text{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}\ ) \quad \subset \quad \mathbb{CH}^5$$

$p \in \mathcal{P}_0$ $\rightsquigarrow$ branched cover $X_p \longrightarrow \mathbb{CP}^1$ with cyclic action σ

$\rightsquigarrow$ $\Lambda(X_p) \cong \Lambda$ integral cohomology with intersection form

$p \in \mathcal{P}_0^{\mathbb{R}}$ $\rightsquigarrow$ antiholomorphic involution $X_p \xrightarrow{\kappa_p} X_p$

induced by complex conjugation $\mathbb{CP}^1 \xrightarrow{\kappa} \mathbb{CP}^1$

$\rightsquigarrow$ involutive anti-isometry (IAI) $\Lambda(X_p) \xrightarrow{\kappa_p^*} \Lambda(X_p)$

$\rightsquigarrow$ IAI $\Lambda \xrightarrow{\chi_p} \Lambda,$ which fixes the period of p

A-C-T construction of $\mathcal{M}_s^{\mathbb{R}}$: Re-assemble the $\mathbb{RH}^5$'s "appropriately."

$\mathbb{P}\mathrm{IAI}(\Lambda)^{\mathbb{R}}$ **and** $\mathbb{P}\mathrm{IAI}(\Lambda)^{\text{antip}}$

$$\mathbb{P}\mathrm{IAI}(\Lambda) \quad = \quad \mathbb{P}\mathrm{IAI}(\Lambda)^{\mathbb{R}} \bigsqcup \mathbb{P}\mathrm{IAI}(\Lambda)^{\text{antip}}$$

$$\frac{\mathbb{P}\mathrm{IAI}(\Lambda)}{\mathbb{P}\mathrm{Isom}(\Lambda)} \quad = \quad \underbrace{\frac{\mathbb{P}\mathrm{IAI}(\Lambda)^{\mathbb{R}}}{\mathbb{P}\mathrm{Isom}(\Lambda)}}_{5\text{ elements}} \bigsqcup \underbrace{\frac{\mathbb{P}\mathrm{IAI}(\Lambda)^{\text{antip}}}{\mathbb{P}\mathrm{Isom}(\Lambda)}}_{1\text{ element}}$$

$$\frac{\mathbb{P}\mathrm{IAI}(\Lambda)^{\mathbb{R}}}{\mathbb{P}\mathrm{Isom}(\Lambda)} \quad \leftrightarrow \quad \left\{ \text{ connected components of } \mathcal{M}_0^{\mathbb{R}} \right\}$$

$\mathcal{M}_0^{\mathbb{R}}$ and Its Connected Components

$$\begin{aligned}
\mathcal{M}_0^{\mathbb{R}} &\cong \mathbb{P}\mathsf{Isom}(\Lambda) \Bigg\backslash \left(\bigsqcup_{[\chi]\in\mathbb{P}\mathsf{IAI}(\Lambda)^{\mathbb{R}}} \left(\mathbb{RH}^5_{[\chi]} - \mathcal{H}\right)\right) \\
&\cong \bigsqcup_{k=0}^{4} \mathsf{Stab}_{\mathbb{P}\mathsf{Isom}(\Lambda)}\left(\mathbb{RH}^5_{[\chi_k]}\right) \Big\backslash \left(\mathbb{RH}^5_{[\chi_k]} - \mathcal{H}\right) \\
&\cong \bigsqcup_{k=0}^{4} \mathcal{M}_0^{\mathbb{R},k},
\end{aligned}$$

where $\mathbb{P}\mathsf{IAI}(\Lambda)^{\mathbb{R}}/\mathbb{P}\mathsf{Isom}(\Lambda) = \{\ [\chi_0],\ \ldots,\ [\chi_4]\ \}\,.$

Enumerating Elements of $\mathbb{P}\mathsf{IAI}(\Lambda)^{\mathbb{R}}$ & $\mathbb{P}\mathsf{IAI}(\Lambda)^{\mathsf{antip}}$

$\mathbb{P}\mathsf{Isom}(\Lambda)\backslash\mathbb{CH}^5$ has only one cusp. Λ has only one $\mathsf{rank}_{\mathbb{Z}[\,\mathbf{i}\,]}$-one $\mathbb{P}\mathsf{Isom}(\Lambda)$-conjugacy class of null sublattices. Λ has only one $\mathbb{P}\mathsf{Isom}(\Lambda)$-conjugacy class of primitive null vectors.

Every element of $\mathsf{IAI}(\Lambda)$ fixes a primitive null vector.

Fix a primitive null vector $\mathbf{e}_6 \in \Lambda$.

Then, every element of $\mathsf{IAI}(\Lambda)$ has a $\mathbb{P}\mathsf{Isom}(\Lambda)$-conjugate that fixes $\mathbf{e}_6$.

A complete list (but with redundancy) of $\mathsf{Isom}(\Lambda, \mathbf{e}_6)$-conjugacy classes of $\mathbf{e}_6$-preserving involutive anti-isometries is computable:

$$\phi_{\mathsf{I}}, \quad \phi_{\mathsf{I}'}, \quad \phi_{\mathsf{II}}, \quad \phi_{\mathsf{II}'}, \quad \ldots \ , \quad \phi_{\mathsf{VII}'}.$$

The above is also a complete list (with redundancy) of representatives of elements of $\mathbb{P}\mathsf{IAI}(\Lambda)/\mathbb{P}\mathsf{Isom}(\Lambda)$.

Each “primed and unprimed” pair has isometric fixed lattices.

Identifying "Topological Types" of $\phi_{\mathsf{I}}, \phi_{\mathsf{I}'}, \ldots, \phi_{\mathsf{VII}}, \phi_{\mathsf{VII}'}$

O $\left(\frac{\Lambda}{(1+\mathbf{i})\Lambda}\right) \cong S_8$. The monondromy group $\mathbb{P}\mathsf{Isom}(\Lambda)$ is generated by "half turns" of pairs of roots. The S_8 above is realized as the permutation group of the 8 distinct roots of a smooth octic.

Compute induced action of each of $\phi_{\mathsf{I}}, \phi_{\mathsf{I}'}, \ldots, \phi_{\mathsf{VII}}, \phi_{\mathsf{VII}'}$:

IAI	cycle structure	type	re-labeling
ϕ_{I} , $\phi_{\mathsf{I}'}$	$(1)(2)(3)(4)(56)(78)$	2	χ_2
ϕ_{II} , $\phi_{\mathsf{II}'}$	$(1)(2)(3)(4)(5)(6)(78)$	1	χ_1
ϕ_{III} , $\phi_{\mathsf{III}'}$	$(1)(2)(34)(56)(78)$	3	χ_3
ϕ_{IV} , $\phi_{\mathsf{IV}'}$	$(1)(2)(3)(4)(5)(6)(7)(8)$	0	χ_0
ϕ_{V} , $\phi_{\mathsf{V}'}$	same as ϕ_{I} , $\phi_{\mathsf{I}'}$	2	
ϕ_{VI} , $\phi_{\mathsf{VI}'}$	$(12)(34)(56)(78)$	4 or antip	
ϕ_{VII} , $\phi_{\mathsf{VII}'}$	$(12)(34)(56)(78)$	4 or antip	

About $\mathsf{Stab}_{\mathbb{P}\mathsf{Isom}(\Lambda)}\left(\mathbb{RH}^5_{[\chi_k]}\right)$ and Distinguishing ϕ_{VI} & ϕ_{VII}

$$\mathsf{Stab}_{\mathbb{P}\mathsf{Isom}(\Lambda)}\left(\mathbb{RH}^5_{[\chi_k]}\right) = \mathbb{P}\mathsf{Stab}_{\mathsf{Isom}(\Lambda)}\left(\mathsf{Fix}\,\chi_k\right).$$

$\mathsf{Fix}(\chi_k)$ is a $\mathbb{Z}$-lattice; its computation is straightforward. So is that of $\mathbb{P}\mathsf{Stab}_{\mathsf{Isom}(\Lambda)}\left(\mathsf{Fix}\,\chi_k\right)$.

$$\mathsf{Stab}_{\mathsf{Isom}(\Lambda)}\left(\mathsf{Fix}\,\chi_k\right) \subsetneq \mathsf{Isom}\big(\mathsf{Fix}\,\chi_k\big)$$

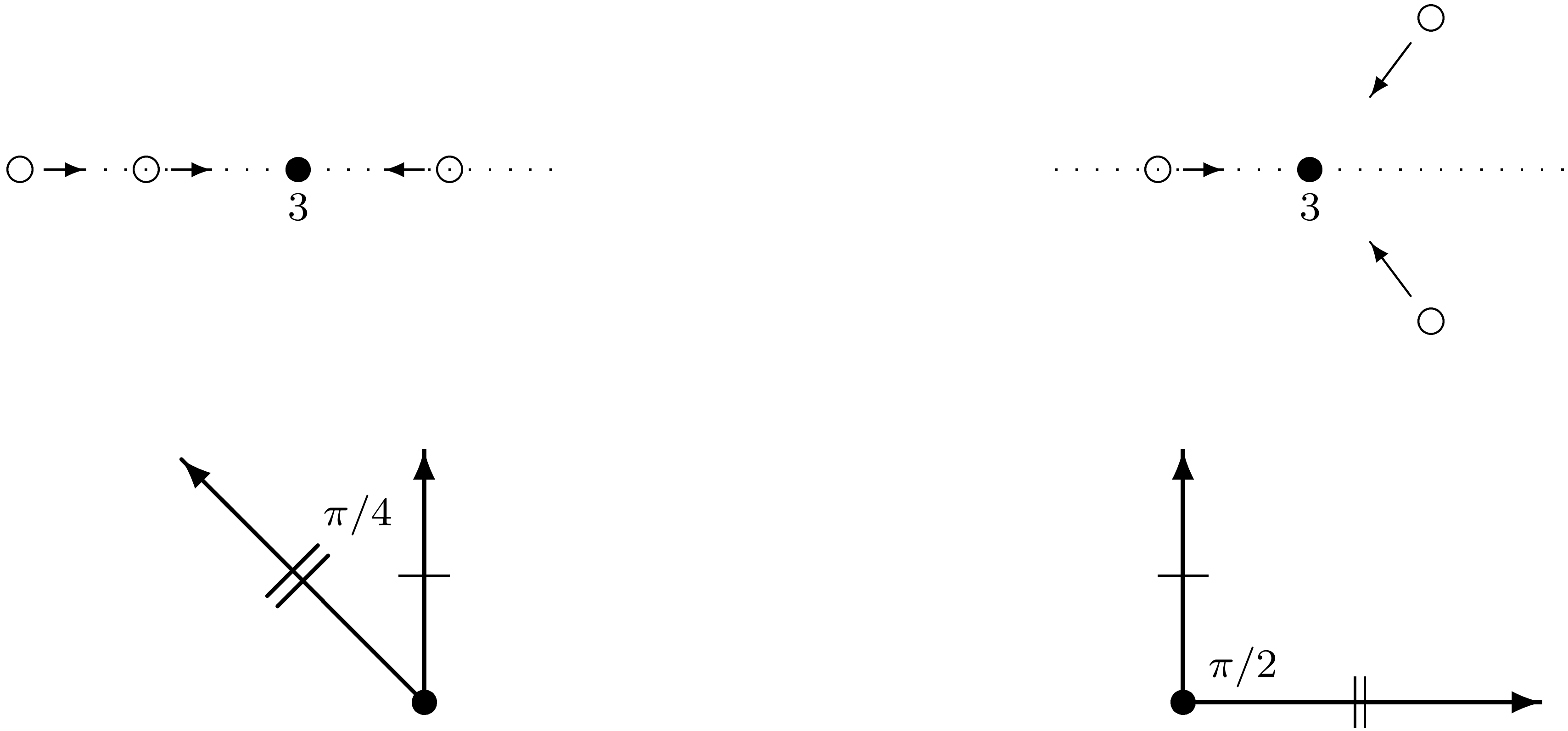
The *Vinberg diagrams* for the following are computed: $\mathsf{Isom}\big(\mathsf{Fix}\,\phi_{\mathsf{VI}}\big)$, $\mathsf{Isom}\big(\mathsf{Fix}\,\phi_{\mathsf{VII}}\big)$, and $\mathsf{Isom}\big(\mathsf{Fix}\,\chi_k\big)$, for $k = 0, 1, 2$. These computations show that each of the above is a reflection group.

The fundamental domain of $\mathsf{Isom}\big(\mathsf{Fix}\,\phi_{\mathsf{VI}}\big)$ “has a discriminant wall”, while that of $\mathsf{Isom}\big(\mathsf{Fix}\,\phi_{\mathsf{VII}}\big)$ does not, which implies

$$\phi_{\mathsf{VI}} = \chi_4 \quad \text{and} \quad \phi_{\mathsf{VII}} = \chi_{\mathsf{antip}}.$$

A-C-T Construction of $\mathcal{M}_s^{\mathbb{R}}$ is NOT a Hyperbolic Orbifold

Two kinds of real smooth 3-point configurations can deform to a real triple point.



vertex angle $= \frac{3\pi}{4} \neq \frac{2\pi}{n}$

THE END

THANK YOU!