

# **The Moduli Space of Real Binary Octics**

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Doctoral Thesis Defense

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Real := defining polynomial has coeff's in  $\mathbb{R}$

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$i$	0	1	2	3	4
# complex conjugate pairs	0	1	2	3	4
# real points	8	6	4	2	0

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1.  $\mathcal{M}_0^{\mathbb{R},i} \cong \mathbb{P}\Gamma_i^{\mathbb{R}} \setminus (\mathbb{R}H^5 - \mathcal{H})$ ,  $i = 0, \dots, 4$ , with  $\mathbb{P}\Gamma_i^{\mathbb{R}}$  explicitly given. They are arithmetic groups.

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2.  $\mathbb{P}\Gamma_0^{\mathbb{R}}, \mathbb{P}\Gamma_1^{\mathbb{R}}, \mathbb{P}\Gamma_2^{\mathbb{R}}, \mathbb{P}\Gamma_4^{\mathbb{R}}$  are finite-index subgroups of discrete reflection groups in  $\text{Isom}(\mathbb{R}\mathbb{H}^5)$ .

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3. The Allcock-Carlson-Toledo construction of  $\mathcal{M}_s^{\mathbb{R}}$  is not a hyperbolic orbifold. (In contrast with the cases of real cubic surfaces and real binary sextics.)

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$$\mathcal{H} := \bigcup \left\{ \mathbb{C}\mathbb{H}(r^\perp) \subset \mathbb{C}\mathbb{H}^5 \mid r \in \Lambda, \langle r, r \rangle = -2 \right\}$$

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A-C-T construction of  $\mathcal{M}_s^{\mathbb{R}}$ : Re-assemble the  $\text{RH}^5$ 's  
 “appropriately.”

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$$\frac{\mathbb{P}IAI(\Lambda)^{\mathbb{R}}}{\mathbb{P}Isom(\Lambda)} \leftrightarrow \left\{ \text{connected components of } \mathcal{M}_0^{\mathbb{R}} \right\}$$

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 &\cong \bigsqcup_{k=0}^4 \text{Stab}_{\mathbb{P}\text{Isom}(\Lambda)}(\mathbb{R}H_{[\chi_k]}^5) \setminus \left( \mathbb{R}H_{[\chi_k]}^5 - \mathcal{H} \right)
 \end{aligned}$$

# $\mathcal{M}_0^{\mathbb{R}}$ and Its Connected Components

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Fix a primitive null vector  $e_6 \in \Lambda$ .

Then, every element of  $|\Lambda|(\Lambda)$  has a  $\mathbb{P}\text{Isom}(\Lambda)$ -conjugate that fixes  $e_6$ .

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A complete list (but with redundancy) of  $\text{Isom}(\Lambda, e_6)$ -conjugacy classes of  $e_6$ -preserving involutive anti-isometries is computable:

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Every element of  $|\text{AI}(\Lambda)$  fixes a primitive null vector.

Fix a primitive null vector  $e_6 \in \Lambda$ .

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Each “primed and unprimed” pair has isometric fixed lattices.

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IAI	cycle structure	type	re-labeling
$\phi_I, \phi_{I'}$	(1)(2)(3)(4)(56)(78)	2	$\chi_2$
$\phi_{II}, \phi_{II'}$	(1)(2)(3)(4)(5)(6)(78)	1	$\chi_1$
$\phi_{III}, \phi_{III'}$	(1)(2)(34)(56)(78)	3	$\chi_3$
$\phi_{IV}, \phi_{IV'}$	(1)(2)(3)(4)(5)(6)(7)(8)	0	$\chi_0$
$\phi_V, \phi_{V'}$	same as $\phi_I, \phi_{I'}$	2	
$\phi_{VI}, \phi_{VI'}$	(12)(34)(56)(78)	4 or antip	
$\phi_{VII}, \phi_{VII'}$	(12)(34)(56)(78)	4 or antip	

# **About $\text{Stab}_{\mathbb{P}\text{Isom}(\Lambda)} \left( \mathbb{R}\text{H}_{[\chi^k]}^5 \right)$ and Distinguishing $\phi_{\text{VI}}$ & $\phi_{\text{VII}}$**

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$$\phi_{\text{VI}} = \chi_4 \quad \text{and} \quad \phi_{\text{VII}} = \chi_{\text{antip}}.$$

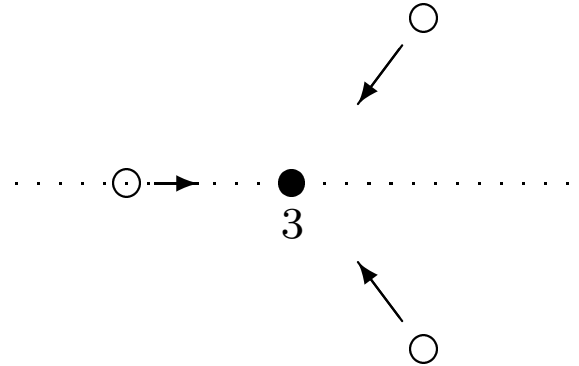
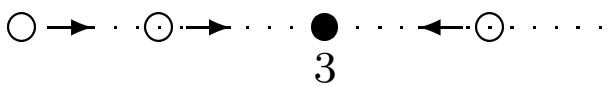
# A-C-T Construction of $\mathcal{M}_g^{\mathbb{R}}$ is NOT a Hyperbolic Orbifold

# A-C-T Construction of $\mathcal{M}_s^{\mathbb{R}}$ is NOT a Hyperbolic Orbifold

Two kinds of real smooth 3-point configurations can deform to a real triple point.

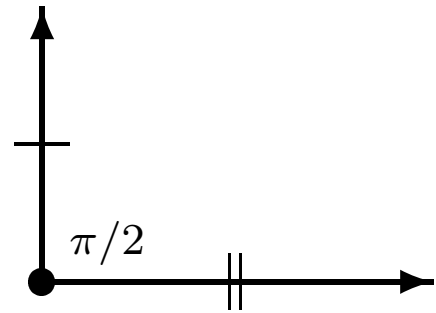
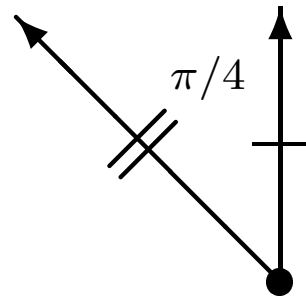
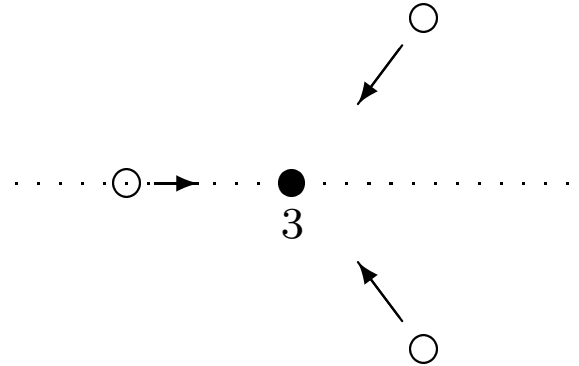
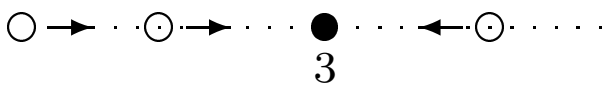
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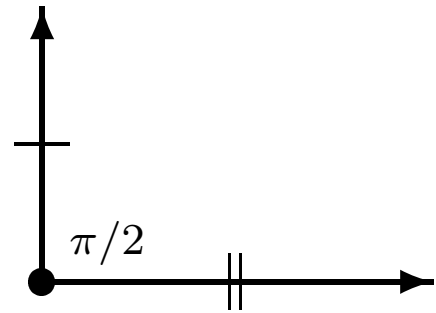
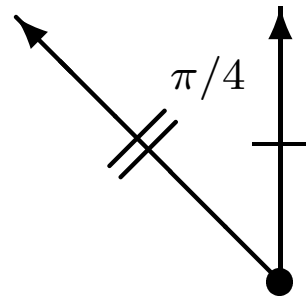
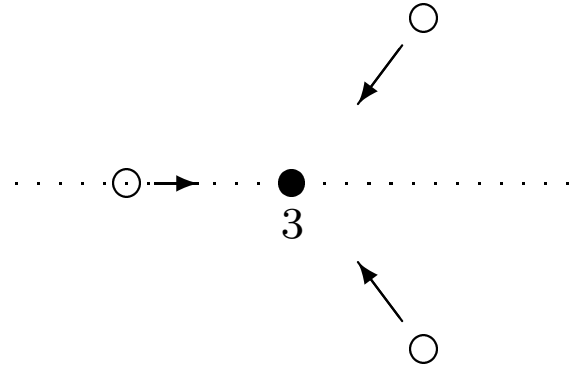
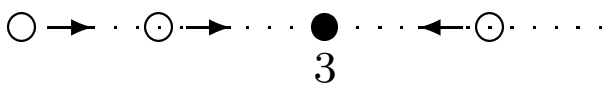
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vertex angle =  $\frac{3\pi}{4} \neq \frac{2\pi}{n}$

**THE END**

**THANK YOU!**