Contributions to the Study of the Validity of Huygens' Principle for the Non-self-adjoint Scalar Wave Equation on Petrov Type D Spacetimes

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The Scalar Wave Equation on a Curved Spacetime

$$P\,u := \Box\,u + A^a\,\nabla_a\,u + B\,u = f$$

- u is the unknown scalar field.
- The differential operator $\Box$ is defined by:

$$\Box := g^{ab}\nabla_a\nabla_b = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^a}\left(\sqrt{|g|}\,g^{ab}\,\frac{\partial}{\partial x^b}\right)$$

- A^a is a given smooth vector field and B, f are given smooth scalar fields.
- The *adjoint operator* ${}^tP$ of P is defined as:

$$\begin{aligned} {}^tP[v] &:= \Box\,v - \nabla_a(A^a\,v) + B\,v \\ &= \Box\,v - A^a\nabla_a v + (B - \mathrm{div}A)\,v \end{aligned}$$

- P is said to be *self-adjoint* if

$${}^tP = P.$$

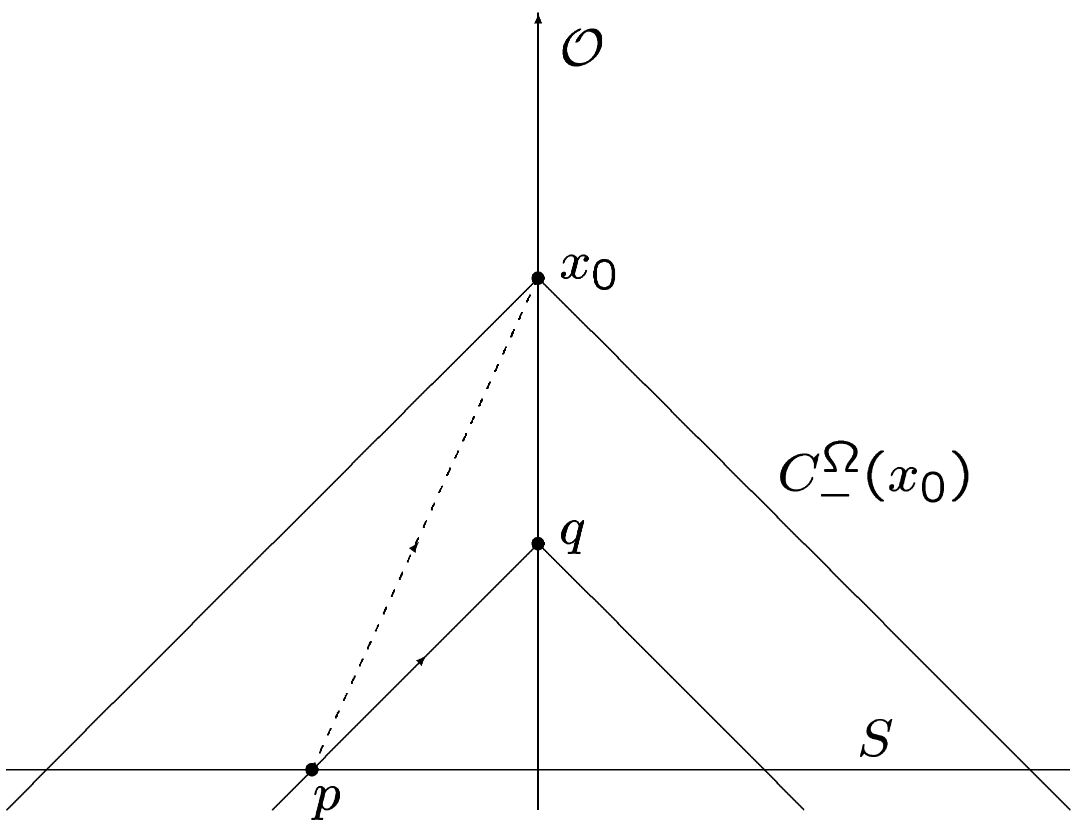
  If ${}^tP \neq P$, then P is said to be *non-self-adjoint*. Clearly, P is self-adjoint if and only if $A^a = 0$.

Huygens' Principle in Words

(in the sense of Hadamard's Minor Premise)

The signal generated by a disturbance localized in space and in time travels in a thin "spherical" shell.

Illustration of Huygens' Principle



The dash line indicates "ripples" from the event p on the Cauchy surface S to the event x_0.

Intuitive idea of Huygens' principle:
The solution at x_0 should *not* depend on the Cauchy data in the interior of $J_-^\Omega(x_0) \cap S$.

Examples

- Sound waves and light waves in 3-dimensional space satisfy Huygens' principle.

- 2-dimensional waves in an elastic membrane possess wave tails, which means Huygens' principle does not hold for these waves.

Hadamard proved that for Huygens' principle to hold, the dimension of the spacetime must be even and greater than or equal to 4.

Hadamard's Conjecture

Every scalar wave equation on a curved spacetime that satisfies Huygens' principle is "equivalent" to the ordinary wave equation in Minkowski spacetime.

Hadamard's Conjecture is now known to be false. Günther (1965) established that any exact plane wave spacetime is a non-conformally flat spacetime on which the self-adjoint scalar wave equation satisfies Huygens principle.

Hadamard's Problem

Determine all equivalence classes of Huygens' scalar wave operators on spacetimes (modulo the trivial transformations).

Three Known Facts

- The conformally invariant scalar wave equation*

$$\Box\, u + \frac{R}{6}\, u = 0 \qquad (1)$$

  satisfies Huygens' principle on any conformally flat spacetime and also on any spacetime conformally related to an exact plane wave spacetime, the metric of which has the form (in Ehlers-Kundt coordinates):

$$\mathrm{d}s^2 = 2\mathrm{d}v \left\{ \mathrm{d}u + \left(D(v) z^2 + \bar{D}(v) \bar{z}^2 + e(v) z \bar{z} \right) \mathrm{d}v \right\} - 2\mathrm{d}z \mathrm{d}\bar{z}.$$

- These are the only known spacetimes on which Huygens' principle is valid for (1).

- These are the only possible conformally empty† spacetimes for which Huygens' principle is valid for (1). [McLenaghan, 1969]

*Any self-adjoint equation satisfying Huygens' principle must have the form (1).

†conformally related to a spacetime with $R_{ab} = 0$

Conjecture (Carminati & McLenaghan)

- Every non-conformally flat spacetime on which Huygens' principle is valid for the conformally invariant scalar wave equation is conformally related to an exact plane wave spacetime.

- Every non-self-adjoint scalar wave equation that satisfies Huygens' principle is equivalent to a self-adjoint equation either on a conformally flat spacetime or on an exact plane wave spacetime.

Partial Results

	I	II	D	III	N	0
conformally invariant			$\times$	$\times$	e.p.w.	$\surd$
non-self-adjoint			?	$\times$	e.p.w.	$\surd$

The (Local) Cauchy Problem

$$\begin{aligned} P\,u &= f(x) \quad \text{in } \Omega \\ u &= g(x) \quad \text{on } S \\ \frac{\partial u}{\partial n} &= h(x) \quad \text{on } S \end{aligned}$$

- Ω is a *causal* domain of the underlying spacetime.
- S, called the Cauchy surface, is a space-like past-compact hypersurface of Ω.
- n is the unit normal on S.
- The smooth functions $g(x)$ and $h(x)$ are the Cauchy data on S.

Theorem (Existence and Uniqueness)
Let S be a past-compact space-like hypersurface such that $\partial J_{+}^{\Omega}(S) = S$. Suppose that $f \in C^\infty(\Omega)$ and $g, h \in C^\infty(S)$. Then the Cauchy problem has a unique solution $u \in C^\infty(J_{+}^{\Omega}(S))$.

Mathematical Formulation

The operator $P := \Box + A^a \nabla_a + B$ is said to be a *forward Huygens'* operator on Ω if for every local Cauchy problem with differential operator P and Cauchy surface $S \subset \Omega$, the support of the solution at $x_0 \in \Omega$ is contained in $C_-^\Omega(x_0) \cap S$, for every $x_0 \in \Omega$.

Hadamard's Criterion

$$P[U] = 0, \qquad \text{on } C_-^\Omega(x_0), \text{ for every } x_0 \in \Omega,$$

where

$$U(x_0, x) := \exp\left\{-\frac{1}{4}\int_0^{s(x)} (\Box\Gamma + A^a \nabla_a \Gamma - 8)\frac{\mathrm{d}t}{t}\right\}.$$

Equivalently,

$$\sigma := \frac{P[U]}{U} = 0, \qquad \text{on } C_-^\Omega(x_0), \text{ for every } x_0 \in \Omega.$$

The Necessary Conditions

It can be shown that

$$\begin{array}{c}\sigma = 0\\ \text{on } C_-^\Omega(x_0) \text{ or } C_+^\Omega(x_0)\end{array} \implies \begin{array}{c}\mathsf{TS}[\,\overset{\circ}{\sigma}_{;a_1\cdots a_m}\,] = 0,\\ \text{for every integer } m \geq 0.\end{array}$$

$$\begin{array}{rcl} H_{ab} & := & A_{[a,b]} \\ L_{ab} & := & -R_{ab} + \frac{R}{6} g_{ab} \end{array} \qquad \begin{array}{rcl} C_{abcd} & := & R_{abcd} - 2g_{[a[d}L_{b]c]} \\ S_{abc} & := & L_{a[b;c]} \end{array}$$

The 0-index to 5-index Conditions

$$\begin{aligned}
0 &= B - \frac{1}{2}A^k{}_{;k} - \frac{1}{4}A_k A^k + \frac{R}{6} \\
0 &= H^k{}_{a;k} \\
0 &= S_{abk;}{}^k - \frac{1}{2}C^k{}_{ab}{}^l L_{kl} + 5\left(H_{ak}\,H_b{}^k - \frac{1}{4}\,g_{ab}\,H_{kl}H^{kl}\right) \\
0 &= \mathsf{TS}\left(3S_{abk}H^k{}_c + C^k{}_{ab}{}^l H_{ck;l}\right) \\
0 &= \left\{ \begin{array}{l} \mathsf{TS}\big(3C_{kabl;m}C^k{}_{cd}{}^l{}_;{}^m + 8C^k{}_{ab}{}^l{}_{;c}S_{kld} + 40S_{ab}{}^k S_{cdk} \\ -8C^k{}_{ab}{}^l S_{klc;d} - 24C^k{}_{ab}{}^l S_{cdk;l} + 4C^k{}_{ab}{}^l C_l{}^m{}_{ck} L_{dm} \\ +12C^k{}_{ab}{}^l C^m{}_{cdl} L_{km} + 12H_{ka;bc}H^k{}_d - 16H_{ka;b}H^k{}_{c;d} \\ -84H^k{}_a C_{kbcl} H^l{}_d - 18H_{ka}H^k{}_b L_{cd}\big) \end{array} \right. \\
0 &= \left\{ \begin{array}{l} \mathsf{TS}\big(36C_k{}^{ab}{}_l C_{lcdm;k} H^m{}_e - 6C^k{}_{ab}{}^l{}_{;c} C_{lde}{}^m H_{km} \\ -138S_{ab}{}^k C_{kcdl} H^l{}_e + 6S_{abk}H^k{}_{c;de} - 24S_{abk;c}H^k{}_{d;e} \\ +12C^k{}_{ab}{}^l L_{kc} H_{ld;e} - 9C^k{}_{ab}{}^l{}_{;c} L_{kd} H_{le} - 9S_{abk} L_{cd} H^k{}_e\big) \end{array} \right.
\end{aligned}$$

The Necessary Conditions in Spinor Form

$$
\begin{aligned}
H_{ab} &\longleftrightarrow \phi_{AB}\bar{\epsilon}_{\dot{A}\dot{B}} + \bar{\phi}_{\dot{A}\dot{B}}\epsilon_{AB} \\
L_{ab} &\longleftrightarrow 2\left(\Phi_{AB\dot{A}\dot{B}} - \Lambda\epsilon_{AB}\bar{\epsilon}_{\dot{A}\dot{B}}\right) \\
S_{abc} &\longleftrightarrow \Psi^{D}{}_{ABC;D\dot{A}}\,\bar{\epsilon}_{\dot{C}\dot{B}} + \bar{\Psi}^{\dot{D}}{}_{\dot{A}\dot{B}\dot{C};\dot{D}A}\,\epsilon_{CB} \\
C_{abcd} &\longleftrightarrow -\Psi_{ABCD}\bar{\epsilon}_{\dot{A}\dot{B}}\bar{\epsilon}_{\dot{C}\dot{D}} - \bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\epsilon_{AB}\epsilon_{CD}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \phi_{AK;}{}^{K}{}_{\dot{A}} \\
0 &= \begin{cases} \Psi_{ABKL;}{}^{K}{}_{\dot{A}}{}^{L}{}_{\dot{B}} + \bar{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L};}{}^{\dot{K}}{}_{A}{}^{\dot{L}}{}_{B} \\ \Psi_{AB}{}^{KL}\Phi_{KL\dot{A}\dot{B}} + \bar{\Psi}_{\dot{A}\dot{B}}{}^{\dot{K}\dot{L}}\bar{\Phi}_{AB\dot{K}\dot{L}} + \\ 10\,\phi_{AB}\bar{\phi}_{\dot{A}\dot{B}} \end{cases} \\
0 &= \begin{cases} 3\,\Psi_{ABCK;}{}^{K}{}_{(\dot{A}}\bar{\phi}_{\dot{B}\dot{C})} + 3\,\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{K};}{}^{\dot{K}}{}_{(A}\phi_{BC)} \\ -\Psi_{ABC}{}^{K}\,\bar{\phi}_{(\dot{A}\dot{B};\dot{C})K} - \bar{\Psi}_{\dot{A}\dot{B}\dot{C}}{}^{\dot{K}}\,\phi_{(AB;C)\dot{K}} \end{cases}
\end{aligned}
$$

and more . . .

General Methodology

- The MAPLE `NPspinor` package is used to convert the spinor-form necessary equations (1- to 5-index conditions) to dyad form.

- The different Petrov types are treated separately to simplify the dyad-form equations. For example, adopting a spinor dyad canonical to an underlying type D spacetime renders
$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0.$

- Make use of conformal freedom and dyad freedom to further simplify the dyad-form necessary equations.

- The Newman-Penrose field equations, the Bianchi identities, and the commutation relations provide further conditions.

Exploiting the Alignment between the Maxwell and Weyl Principal Spinors

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The Maxwell spinor:

$$A_{[a;b]} =: H_{ab} \longleftrightarrow \phi_{AB}\bar{\epsilon}_{\dot{A}\dot{B}} + \overline{\phi}_{\dot{A}\dot{B}}\epsilon_{AB}$$

ϕ_{AB} is symmetric and thus can be decomposed into principal spinors:

$$\phi_{AB} = \xi_{(A}\,\zeta_{B)}$$

We express ϕ_{AB} with respect to a spinor dyad $\{o_A, \iota_B\}$ canonical to the Weyl spinor as follows:

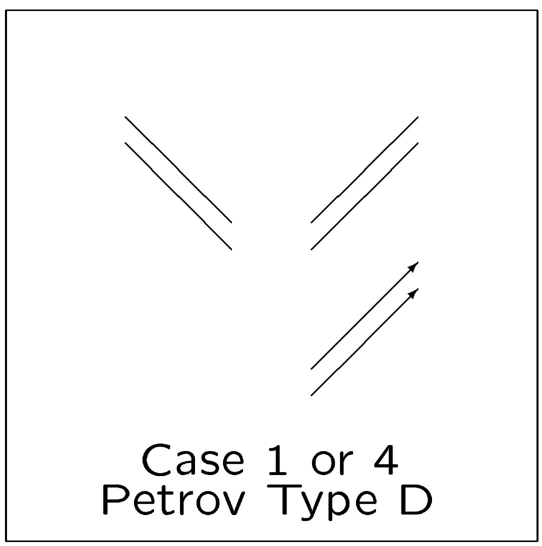
$$\phi_{AB} = \phi_0\,\iota_A \otimes \iota_B - 2\,\phi_1\,o_{(A} \otimes \iota_{B)} + \phi_2\,o_A \otimes o_B,$$

The following table shows the seven alignment possibilities:

Case	ϕ_0	ϕ_1	ϕ_2	$\phi_{AB} \propto$	Status
0	0	0	0	0	
1	0	0	N	$o_A o_B$	×
2	0	N	0	$o_{(A}\iota_{B)}$	?
3	0	N	N		
4	N	0	0	$\iota_A \iota_B$	×
5	N	0	N		
6	N	N	0		
7	N	N	N		

Proposition 1

Let $P := \Box + A^a \nabla_a + B$ be a non-self-adjoint scalar wave operator on a Petrov type D space-time. Then, for P, Huygens' principle and Case 4 (or equivalently, Case 1) are incompatible.



- Dyad freedom: $\phi_0 = \Psi_2$.
  Conformal freedom: $\Psi_2 \overline{\Psi}_2 \equiv 1$.

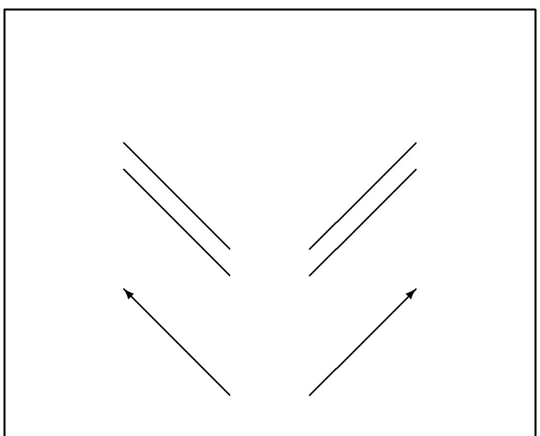
- Dyad equations simplify to polynomial systems in π, α, τ and their conjugates. Gröbner basis methods and the MAPLE function `gsolve()` were used.

Proposition 2

Let $P := \Box + A^a \nabla_a + B$ be a non-self-adjoint scalar wave operator on a Petrov type D spacetime. If P belongs to Case 2, then

- the principal null congruences of the Weyl tensor are geodesic and shear-free, and
- there exists a conformal gauge (in which $\Psi_2 \overline{\Psi}_2 \equiv 1$) such that

$$\bar{\rho} = -\rho, \qquad \bar{\mu} = \mu, \qquad \tau = \bar{\pi}.$$



Case 2
Petrov Type D

Proposition 3 [McLenaghan]

On any symmetric type D spacetime, there exist non-self-adjoint scalar wave operators that satisfy the 0-index to 5-index necessary conditions for Huygens' principle (i.e. all the necessary conditions that have been computed for the non-self-adjoint scalar wave equation).

This suggests symmetric type D spacetimes may allow counter-examples of the Carminati-McLenaghan conjecture.