The Validity of Huygens' Principle for the Scalar Wave Equation on Curved Spacetimes

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Historical Background

In 1678, the Dutch mathematician, astronmer, and physicist Christiaan Huygens published his *Treatise on Light*, in which he deduced the laws of geometrical optics based on a number of assumptions on the propagation of light waves. These assumptions were later referred to as Huygens' principle.

There had been confusion as to the correct mathematical formulation of Huygens' principle, until 1923 when Jacques Hadamard clarified the question in his *Lectures on Cauchy's Problem in Linear Partial Differential Equations* by stating Huygens' principle in the form of a syllogism.

Hadamard's Syllogism

- **Major Premise**
  The action of phenomena produced at the instant $t = 0$ on the state of matter at the later time $t = t_0$ takes place by the mediation of every intermediate instant $t = t'$, i.e. (assuming $0 < t' < t_0$), in order to find out what takes place for $t = t_0$, we can deduce from the state at $t = 0$ the state at $t = t'$ and from the latter, the required state at $t = t_0$.

- **Minor Premise**
  If, at the instant $t = 0$ — or more precisely throughout the short interval $-\epsilon \leq t \leq 0$ — we produce a luminous disturbance localized in the intermediate neighbourhood of O, the effect of it will be, for $t = t'$, localized in the immediate neighbourhood of the surface of the sphere with center O and radius $\omega t'$: that is, will be localized in a very thin spherical shell with centre O including the aforesaid sphere.

- **Conclusion**
  In order to calculate the effect of our initial luminous phenomenon produced at O at $t = 0$, we may replace it by a proper system of disturbances taking place at $t = t'$ and distributed over the surface of the sphere with centre O and radius $\omega t'$.

The Scalar Wave Equation on a Curved Spacetime*

$$P\,u := \Box\,u + A^a\,\nabla_a\,u + B\,u = f(x) \qquad (1)$$

- u is the unknown scalar field.
- The differential operator $\Box$ is defined by:

$$\Box := g^{ab}\nabla_a\nabla_b = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^a}\left(\sqrt{|g|}\,g^{ab}\,\frac{\partial}{\partial x^b}\right)$$

- A^a is a given vector field and B is a given scalar field.
- The adjoint operator ${}^tP$ of P is defined as:

$$\begin{aligned} {}^tP[v] &:= \Box\,v - \nabla_a(A^a\,v) + B\,v \\ &= \Box\,v - A^a\nabla_a v + (B - \text{div}A)\,v \end{aligned}$$

*a Lorentzian manifold:

Examples

- Sound waves and light waves in 3-dimensional space satisfy Huygens' principle.

- 2-dimensional waves in an elastic membrane possess wave tails, which means Huygens' principle does not hold for these waves.

  Hadamard proved that for Huygens' principle to hold, the dimension of the spacetime must be even and greater than or equal to 4.

Hadamard's Conjecture

Every equation of the form (1) that satisfies Huygens' principle is "equivalent" to the ordinary wave equation in Minkowski spacetime.

Hadamard's Conjecture is now known to be false. In fact, Stellmacher (1955) constructed counter-examples to Hadamard's conjecture for all even dimensions.

Günther (1965) established that any exact plane wave spacetime is a non-conformally flat spacetime on which the self-adjoint scalar wave equation satisfies Huygens principle.

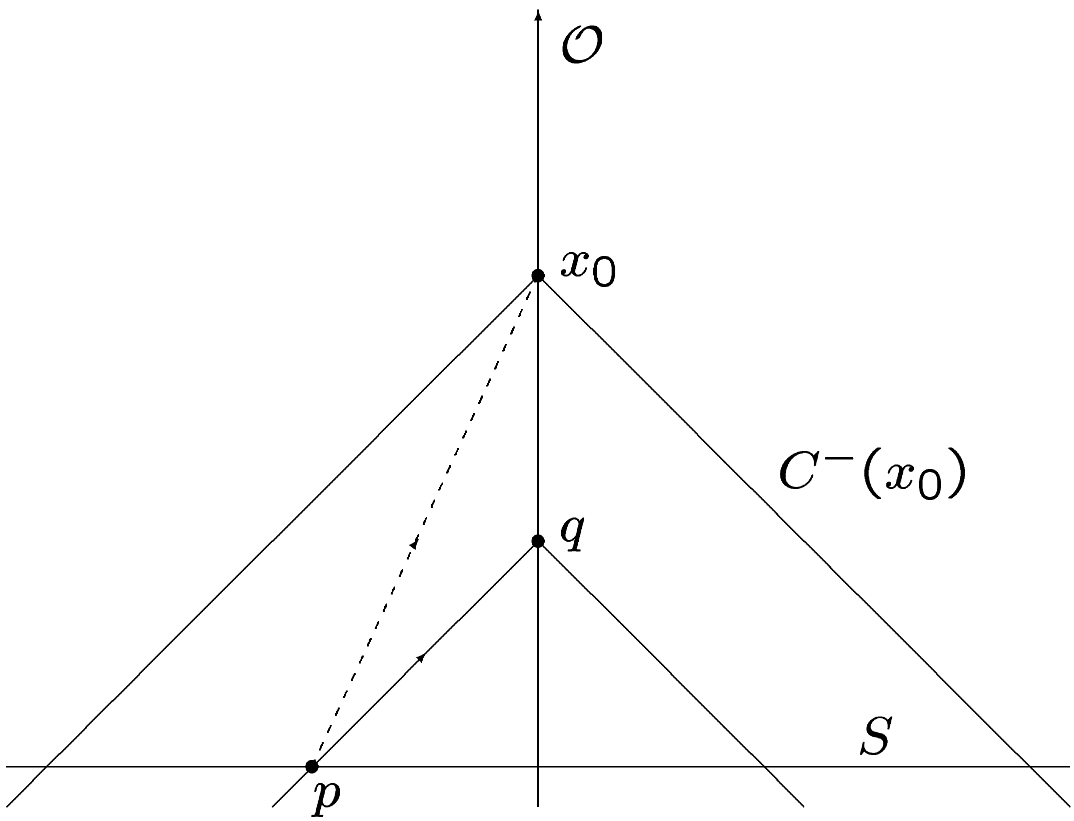
The (Local) Cauchy Problem

$$\begin{aligned} P\,u &= f(x) \quad \text{in } \Omega \\ u &= g(x) \quad \text{on } S \\ \frac{\partial u}{\partial n} &= h(x) \quad \text{on } S \end{aligned}$$

- Ω is a *causal* domain of the underlying spacetime.
- S, called the Cauchy surface, is a space-like hypersurface of Ω.
- n is the unit normal on S.
- The functions $g(x)$ and $h(x)$ are the Cauchy data on S.

Theorem (Existence and Uniqueness)
Let S be a past-compact space-like hypersurface such that $\partial J^+(S) = S$. Suppose that $f \in C^\infty(\Omega)$ and $g, h \in C^\infty(S)$. Then the Cauchy problem has a unique solution $u \in C^\infty(J^+(S))$.

Illustration of Huygens' Principle



The dash line indicates "ripples" from the event p on the Cauchy surface S to the event x_0.

Intuitive idea of Huygens' principle:
The solution at x_0 should *not* depend on the Cauchy data in $D^-(x_0) \cap S$.

Theorem (Representation Formula)

Let Ω be a causal domain and $P := \Box + A^a \nabla_a + B$ a scalar wave operator on $C^2(\Omega)$. Let S be a past-compact space-like hypersurface such that $\partial J^+(S) = S$.

If $u \in C^\infty(\Omega)$, then for each $x_0 \in J^+(S) \backslash S$, we have

$$u(x_0) = u^{(1)}(x_0) + u^{(2)}(x_0) + u^{(3)}(x_0),$$

where

$$u^{(1)}(x_0) := \frac{1}{2\pi} \left(\int_{C^-(x_0) \cap J^+(S)} U P(u) \mu_\Gamma + \int_{J^-(x_0) \cap J^+(S)} V^+ P(u) \mu \right)$$

$$u^{(2)}(x_0) := \frac{1}{2\pi} \int_{J^-(x_0) \cap S} * \left(V^+ \nabla(u) - u \nabla V^+ + (u\, V^+) A^a \right)$$

$$u^{(3)}(x_0) := \frac{1}{2\pi} \int_{C^-(x_0) \cap S} (\cdots)$$

$U \in C^\infty(\Omega)$ is defined by:

$$U(x_0, x) := \exp \left\{ -\frac{1}{4} \int_0^{s(x)} (\Box \Gamma + A^a \nabla_a \Gamma - 8) \frac{\mathrm{d}t}{t} \right\}$$

$V^+(x_0, x) \in C^\infty(D^+(x_0))$ is the solution to the following initial value problem:

$$\begin{cases} P[V^+] &= 0, & \text{on } D^+(x_0), \text{ and} \\ V^+(x_0, x) &= \frac{U(x_0,x)}{s} \int_0^{s(x)} \frac{P[U]}{U} \mathrm{d}t, & \text{when } x \in C^+(x_0). \end{cases}$$

Mathematical Formulation

The operator $P := \Box + A^a \nabla_a + B$ is said to be a Huygens' operator on Ω if for every local Cauchy problem with differential operator P and Cauchy surface $S \subset \Omega$, the support of the solution at $x_0 \in \Omega$ is contained in $C^{\pm}(x_0) \cap S$, for every $x_0 \in \Omega$.

Necessary and Sufficient Conditions

Immediately, a sufficient condition:

$$V^{\pm}(x_0, x) = 0, \qquad \begin{array}{c} \text{for every } x \in J^{\pm}(x_0) \\ \text{for every } x_0 \in \Omega. \end{array}$$

It can be shown that it is also a necessary condition. It has an equivalent form:

$$\sigma := \frac{P[U]}{U} = 0, \qquad \begin{array}{c} \text{on } C^{\pm}(x_0), \\ \text{for every } x_0 \in \Omega. \end{array}$$

The latter is more convenient for computations.

Trivial Transformations

- Conformal Transformations of Spacetime
- Replacing the dependent variable u by $\lambda(x)\,u$, and multiplying $P(\lambda u)$ by $1/\lambda$, where $\lambda(x)$ is a nowhere vanishing function.
- Coordinate Transformations

Theorem
Suppose $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be two spacetimes such that one can be transformed to the other by a combination of transformations of the above types. Let

$$P := \Box + A^a \nabla_a + B$$

be a hyperbolic differential operator on $\mathcal{M}$ with metric principal part and let

$$\tilde{P} := \tilde{\Box} + \tilde{A}^a \tilde{\nabla}_a + \tilde{B}$$

be the corresponding transformed operator on $\tilde{\mathcal{M}}$. Then, P is a Huygens operator if and only if $\tilde{P}$ is a Huygens operator.

Definition
Two spacetimes that can be transformed from one to the other through a combination of the above types of transformations are said to be *equivalent*.

Hadamard's Problem

Hadamard's Problem
Determine all equivalence classes of Huygens' operators modulo the trivial transformations on the set of all second order linear hyperbolic differential operators with metric principal part on a Lorentzian manifold.

Some Known Results

- The self-adjoint equation (conformally invariant scalar wave equation) is valid on any conformally flat spacetime and also on any spacetime conformally related to an exact plane wave spacetime, the metric of which has the form (in Ehlers-Kundt coordinates):

$$\mathrm{d}s^2 = 2\mathrm{d}v\left\{\mathrm{d}u + \left(D(v)z^2 + \bar{D}(v)\bar{z}^2 + e(v)z\bar{z}\right)\mathrm{d}v\right\} - 2\mathrm{d}z\mathrm{d}\bar{z}.$$

- These are the only known spacetimes on which Huygens' principle is valid for the conformally invariant equation.
- These are the only conformally empty spacetimes for which Huygens' principle is valid for the conformally invariant equation. [McLenaghan, 1969]

Conjecture
(Carminati & McLenaghan)

- Every non-conformally flat spacetime on which Huygens' principle is valid for the self-adjoint scalar wave equation is conformally related to an exact plane wave spacetime.

- Every non-self-adjoint scalar wave equation that satisfies Huygens' principle is equivalent to a self-adjoint one in a conformally flat spacetime or in an exact plane wave spacetime.

The 0- to 5-index Necessary Conditions

Assuming the validity of Huygens' principle, one expands the *diffusion kernel* σ as a covariant Taylor expansion about an arbitrary point $x_0 \in \Omega$.

$$\sigma := \frac{PU}{U} = \sigma_{,a}|_{x_0}\, x^a + \mathsf{TS}(\sigma_{;ab})|_{x_0}\, x^a x^b + \cdots$$

$$\begin{array}{rcl} H_{ab} & := & A_{[a,b]} \\ L_{ab} & := & -R_{ab} + \frac{R}{6} g_{ab} \end{array} \qquad \begin{array}{rcl} C_{abcd} & := & R_{abcd} - 2 g_{[a[d} L_{b]c]} \\ S_{abc} & := & L_{a[b;c]} \end{array}$$

The Necessary Conditions

$$\begin{array}{rcl}
0 & = & B - \frac{1}{2} A^k{}_{;k} - \frac{1}{4} A_k A^k + \frac{R}{6} \\
0 & = & H^k{}_{a;k} \\
0 & = & S_{abk;}{}^k - \frac{1}{2} C^k{}_{ab}{}^l L_{kl} + 5 \left(H_{ak}\, H_b{}^k - \frac{1}{4}\, g_{ab}\, H_{kl} H^{kl} \right) \\
0 & = & \mathsf{TS}\left(3 S_{abk} H^k{}_c + C^k{}_{ab}{}^l H_{ck;l} \right) \\
0 & = & \left\{ \begin{array}{l} \mathsf{TS}\big(3 C_{kabl;m} C^k{}_{cd}{}^l{}_;{}^m + 8 C^k{}_{ab}{}^l{}_{;c} S_{kld} + 40 S_{ab}{}^k S_{cdk} \\ -8 C^k{}_{ab}{}^l S_{klc;d} - 24 C^k{}_{ab}{}^l S_{cdk;l} + 4 C^k{}_{ab}{}^l C_l{}^m{}_{ck} L_{dm} \\ +12 C^k{}_{ab}{}^l C^m{}_{cdl} L_{km} + 12 H_{ka;bc} H^k{}_d - 16 H_{ka;b} H^k{}_{c;d} \\ -84 H^k{}_a C_{kbcl} H^l{}_d - 18 H_{ka} H^k{}_b L_{cd} \big) \end{array} \right. \\
0 & = & \left\{ \begin{array}{l} \mathsf{TS}\big(36 C_k{}^{ab}{}_l C_{lcdm;k} H^m{}_e - 6 C^k{}_{ab}{}^l{}_{;c} C_{lde}{}^m H_{km} \\ -138 S_{ab}{}^k C_{kcdl} H^l{}_e + 6 S_{abk} H^k{}_{c;de} - 24 S_{abk;c} H^k{}_{d;e} \\ +12 C^k{}_{ab}{}^l L_{kc} H_{ld;e} - 9 C^k{}_{ab}{}^l{}_{;c} L_{kd} H_{le} - 9 S_{abk} L_{cd} H^k{}_e \big) \end{array} \right.
\end{array}$$

Spinors

- If a Lorentzian 4-manifold $\mathcal{M}$ admits a smooth orthonormal tetrad on all of $\mathcal{M}$, then there exists a *spin structure* on $\mathcal{M}$:
  - We start with the trivial vector bundle $\mathcal{M} \times \mathbb{C}^2$. For each fixed $p \in \mathcal{M}$, $\{p\} \times \mathbb{C}^2$ is naturally a 2-dimensinoal vector space over $\mathbb{C}$, and the dual, conjugate and anti-dual spaces of $\{p\} \times \mathbb{C}^2$ are well-defined.
  - A bijection between the set of all tensor fields and the set of all Hermitian spinor fields on $\mathcal{M}$.
  - A "metric" on the spinor space that is "compatible" with the metric on $\mathcal{M}$.

- If $\mathcal{M}$ admits a spin structure, then there exists a *unique* connection on the spin structure that is "compatible" with the Levi-Civita connection on $\mathcal{M}$.

- Every totally symmetric spinor field can be decomposed into principal spinors.

$$\Phi_{A_1 \cdots A_n} = \overset{1}{\xi}_{(A_1} \cdots \overset{n}{\xi}_{A_n)}$$

- Every spinor ξ_A corresponds to a unique null direction in $\mathcal{M}$.

Petrov Classification

The spinor equivalent of the Weyl tensor C_{abcd} has the form:

$$\Psi_{ABCD}\bar{\epsilon}_{\dot{A}\dot{B}}\bar{\epsilon}_{\dot{C}\dot{D}} + \epsilon_{AB}\epsilon_{CD}\bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}$$

where Ψ_{ABCD} is totally symmetric. Therefore,

$$\Psi_{ABCD} = \alpha_{(A}\,\beta_B\,\gamma_C\,\delta_{D)}$$

where $\alpha_A, \beta_B, \gamma_C, \delta_D$ are the principal spinors of Ψ_{ABCD}.

	Ψ_{ABCD}	Vanishing components
I	$\alpha_{(A}\,\beta_B\,\gamma_C\,\delta_{D)}$	none
II	$\alpha_{(A}\,\alpha_B\,\beta_C\,\gamma_{D)}$	Ψ_0, Ψ_1
D	$\alpha_{(A}\,\alpha_B\,\beta_C\,\beta_{D)}$	Ψ_0, Ψ_1, Ψ_3, Ψ_4
III	$\alpha_{(A}\,\alpha_B\,\alpha_C\,\beta_{D)}$	Ψ_0, Ψ_1, Ψ_2
N	$\alpha_{(A}\,\alpha_B\,\alpha_C\,\alpha_{D)}$	Ψ_0, Ψ_1, Ψ_2, Ψ_3
0	0	all

Due to total symmetry, Ψ_{ABCD} has only the following independent components:

$\Psi_0 := \Psi_{0000},\ \Psi_1 := \Psi_{0001},\ \Psi_2 := \Psi_{0011},\ \Psi_3 := \Psi_{0111},\ \Psi_4 := \Psi_{1111}.$

- Since the Weyl tensor is conformally invariant, Petrov classification is also conformally invariant.

- For each Petrov type, the Weyl spinor components in the third column of the above table vanish with respect to any *canonical* spinor dyad (non-unique) of the Weyl spinor.

The Necessary Conditions in Spinor Form

$$
\begin{array}{lcl}
H_{ab} & \longleftrightarrow & \phi_{AB}\overline{\epsilon}_{\dot{A}\dot{B}} + \overline{\phi}_{\dot{A}\dot{B}}\epsilon_{AB} \\
L_{ab} & \longleftrightarrow & 2\left(\Phi_{AB\dot{A}\dot{B}} - \Lambda\epsilon_{AB}\overline{\epsilon}_{\dot{A}\dot{B}}\right) \\
S_{abc} & \longleftrightarrow & {\Psi^{D}}_{ABC;D\dot{A}}\,\overline{\epsilon}_{\dot{C}\dot{B}} + {\overline{\Psi}^{\dot{D}}}_{\dot{A}\dot{B}\dot{C};\dot{D}A}\,\epsilon_{CB} \\
C_{abcd} & \longleftrightarrow & \Psi_{ABCD}\overline{\epsilon}_{\dot{A}\dot{B}}\overline{\epsilon}_{\dot{C}\dot{D}} + \overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\epsilon_{AB}\epsilon_{CD}
\end{array}
$$

$$
0 = {\phi_{AK;}}^{K}{}_{\dot{A}}
$$

$$
0 = \begin{cases} {\Psi_{ABKL;}}^{K}{}_{\dot{A}}{}^{L}{}_{\dot{B}} + {\overline{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L};}}^{\dot{K}}{}_{A}{}^{\dot{L}}{}_{B} \\ {\Psi_{AB}}^{KL}\Phi_{KL\dot{A}\dot{B}} + {\overline{\Psi}_{\dot{A}\dot{B}}}^{\dot{K}\dot{L}}\overline{\Phi}_{AB\dot{K}\dot{L}} + \\ 10\,\phi_{AB}\overline{\phi}_{\dot{A}\dot{B}} \end{cases}
$$

$$
0 = \begin{cases} 3\,{\Psi_{ABCK;}}^{K}{}_{(\dot{A}}\,\overline{\phi}_{\dot{B}\dot{C})} + 3\,{\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{K};}}^{\dot{K}}{}_{(A}\,\phi_{BC)} \\ -{\Psi_{ABC}}^{K}\,\overline{\phi}_{(\dot{A}\dot{B};\dot{C})K} - {\overline{\Psi}_{\dot{A}\dot{B}\dot{C}}}^{\dot{K}}\,\phi_{(AB;C)\dot{K}} \end{cases}
$$

and more ...

Method of Proof of the Carminati-McLenaghan Conjecture

- Each Petrov Type of spacetimes is treated separately.

- In each case, we perform computations using a spinor dyad canonical to the Weyl spinor of the spacetime, thereby simplifying the "component" equations of the necessary conditions by the vanishing of the respective Weyl spinor components.

- For the non-self-adjoint equation, alignment of the principal spinors of Maxwell's spinor with those of the Weyl spinor may be used to further decompose the problem.

Partial Results re: Hadamard's Problem

	I	II	D	III	N	0
conformally invariant			$\times$	$\times$	e.p.w.	$\surd$
non-self-adjoint			?	$\times$	e.p.w.	$\surd$

- [Mathisson, 1939] The non-self-adjoint scalar wave equation in a conformally flat spacetime satisfies Huygens' principle.

- [Günther, 1965] Huygens' principle is valid for the conformally invariant scalar wave equation on any conformally flat spacetime and also on any spacetime conformally related to an exact plane wave spacetime, the metric of which has the form (in Ehlers-Kundt coordinates):

$$\mathrm{d}s^2 = 2\mathrm{d}v\left\{\mathrm{d}u + \left(D(v)z^2 + \bar{D}(v)\bar{z}^2 + e(v)z\bar{z}\right)\mathrm{d}v\right\} - 2\mathrm{d}z\mathrm{d}\bar{z}$$

- [Carminati & McLenaghan, 1986] The conformally invariant scalar wave equation on a Petrov Type N spacetime satisfies Huygens' principle if and only if the spacetime is conformally related to the *exact plane wave* spacetime.

- [McLenaghan & Walton, 1988] Any non-self-adjoint scalar wave equation satisfies Huygens' principle on a Petrov Type N spacetime if and only if it is equivalent to the wave equation on an exact plane wave spacetime.

- [McLenaghan & Williams, 1990] There are no Petrov Type D spacetimes on which the conformally invariant scalar wave equation satisfies Huygens' principle.

- [Czapor, McLenaghan & Sasse, 1999] There are no Petrov Type III spacetimes on which the scalar wave equation (self-adjoint or not) satisfies Huygens' principle.

Independent Results for the Non-self-adjoint Scalar Wave Equation on a Petrov Type D Spacetime

The Maxwell spinor:

$$A_{[a;b]} =: H_{ab} \longleftrightarrow \phi_{AB}\bar{\epsilon}_{\dot{A}\dot{B}} + \overline{\phi}_{\dot{A}\dot{B}}\epsilon_{AB}$$

ϕ_{AB} is symmetric and thus can be decomposed into principal spinors:

$$\phi_{AB} = \xi_{(A}\,\zeta_{B)}$$

We express ϕ_{AB} with respect to a spinor dyad $\{o_A, \iota_B\}$ canonical to the Weyl spinor as follows:

$$\phi_{AB} = \phi_0\, \iota_A \otimes \iota_B - 2\,\phi_1\, o_{(A} \otimes \iota_{B)} + \phi_2\, o_A \otimes o_B,$$

The following table shows the 7 alignment possibilities:

Case	ϕ_0	ϕ_1	ϕ_2	$\phi_{AB} \propto$	Status
0	0	0	0	0	
1	0	0	N	$o_A o_B$	×
2	0	N	0	$o_{(A}\iota_{B)}$	?!
3	0	N	N		
4	N	0	0	$\iota_A \iota_B$	×
5	N	0	N		
6	N	N	0		
7	N	N	N		

Proposition 1

Let $P := \Box + A^a \nabla_a + B$ be a non-self-adjoint second-order hyperbolic differential operator with metric principal part in a Petrov Type D spacetime.

IF	The Maxwell's spinor[†] associated to P is algebraically special, and its degenerate principal null direction coincides with one of those of the Weyl spinor of the underlying spacetime.
THEN	P is not a Huygens' operator.

$^\dagger A_{[a;b]} =: H_{ab} \longleftrightarrow \phi_{AB}\overline{\epsilon}_{\dot{A}\dot{B}} + \overline{\phi}_{\dot{A}\dot{B}}\epsilon_{AB}$

Sketch Proof of Propostion 1

The set of "component" equations of the necessary conditions consists of:

- 18 Newman-Penrose Field Equations.
- 11 Bianchi Identities (symmetry of Riemann Curvature tensor/spinor)
- "Huygens" equations
  - 4 from 1-index condition
  - 9 from 2-index condition
  - 16 from 3-index condition
  - 25 from 4-index condition
  - 36 from 5-index condition

These are expanded using the Maple package **npspinor**, developed by Czapor and McLenaghan.

The functions that occur in them are:

- The 12 spin coefficients

$$\kappa, \rho, \sigma, \tau, \varepsilon, \alpha, \beta, \gamma, \pi, \lambda, \mu, \nu$$

  and their complex conjugates.

- The Weyl spinor components:

$$\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$$

  and their conjugates.

- The Curvature spinor components

$$\Phi_{00}, \Phi_{01}, \Phi_{10}, \Phi_{11}, \Phi_{12}, \Phi_{21}, \Phi_{22}$$

- The Ricci Scalar: Λ.

- The Maxwell's spinor components

$$\phi_0, \phi_1, \phi_2$$

  and their conjugates.

- The four pfaffian derivatives of each of the above.

$$H_{;A\dot{A}} = \begin{array}{l} D(H)\iota_A \otimes \bar{\iota}_{\dot{A}} + \Delta(H) o_A \otimes \bar{o}_{\dot{A}} \\ -\delta(H)\iota_A \otimes \bar{o}_{\dot{A}} - \bar{\delta}(H) o_A \otimes \bar{\iota}_{\dot{A}} \end{array}$$

Choice of Conformal Gauge and Spinor Dyad

- Start with a spinor dyad $\{o, \iota\}$ that is canonical to our Petrov Type D Weyl spinor.

- Ψ_2, $\bar{\Psi}_2$ are then the only non-vanishing components of the Weyl spinor with respect to this dyad.

- Recall that canonical dyads are not unique.

  In particular, the following dyad is still canonical to the Weyl spinor:

$$o' = e^{w/2}\, o, \qquad \iota' = e^{-w/2}\, \iota,$$

  where w is any complex function on the underlying space-time.

  Transformation Laws of Maxwell's spinor:

$$\phi_0' = e^{w}\, \phi_0 \quad \phi_1' = \phi_1 \quad \phi_2' = e^{-w}\, \phi_2$$

  By hypothesis of **Proposition 1**, $\phi_1 = 0 = \phi_2$, and this fact is preserved under the above dyad transformation as shown by their transformation laws.

We may choose w so that $\Psi_2 = \phi_0$.
Similary, we may furthermore choose a conformal transformation so that $\Psi_2 \bar{\Psi}_2 \equiv 1$.

We have set

$$\phi_0 = \Psi_2 \qquad\qquad \Psi_2\bar{\Psi}_2 \equiv 1$$

Then two of the Maxwell's equations then give:

$$\begin{aligned} D(\phi_2) - \bar{\delta}(\phi_1) &= -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\varepsilon)\phi_2 \\ 0 &= -\lambda\Psi_2 \\ l &= 0 \end{aligned}$$

$$\begin{aligned} \delta(\phi_2) - \Delta(\phi_1) &= -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2 \\ 0 &= -\nu\Psi_2 \\ \nu &= 0 \end{aligned}$$

On the other hand, one of the NP equations says:

$$\begin{aligned} D(\lambda) - \bar{\delta}(\pi) &= \rho\lambda + \bar{\sigma}\mu + \pi^2 + (\alpha - \bar{\beta})\pi \\ &\quad -\nu\bar{\kappa} + (\bar{\epsilon} - 3\varepsilon)\lambda + \Phi_{20} \end{aligned}$$

Hence, $\quad \bar{\delta}(\pi) = -\bar{\sigma}\mu - \pi^2 - \pi\alpha + \pi\bar{\beta} - \Phi_{20}$

In a similar fashion, all of the following pfaffians derivatives are eliminated:

$$\bar{\delta}\pi, \Delta\pi, \bar{\delta}\mu, \Delta\mu, \bar{\delta}\Psi_2, \Delta\Psi_2, D\Psi_2,$$
$$\delta\Psi_2, \Delta\varepsilon, \delta\pi, \Delta\alpha, \Delta\Phi_{11}, \Delta\Phi_{20}, \bar{\delta}\alpha, X\sigma,$$
$$\Delta\Lambda, \Delta\rho, \delta\alpha, V\sigma, \delta\tau, \bar{\delta}\tau, \Delta\rho$$
$$\bar{\delta}\bar{\epsilon}, \bar{\delta}\rho, D\tau, D\kappa, D\pi, D\bar{\pi}, D\bar{\tau}, D\alpha, D\bar{\alpha},$$
$$\Delta\kappa, \delta\varepsilon, \delta\bar{\epsilon}, \delta\rho, \delta\bar{\rho}, \delta\bar{\sigma}, \bar{\delta}\varepsilon, \bar{\delta}\bar{\rho}, \bar{\delta}\sigma$$

When expressions for all the above pfaffian derivatives are substituted back into the set of equations, the majority would "simplify" to algebraic equations.

Furthermore, the following type of factorizations starts to take place (this particular one is from the 5-index condition):

$$-\Psi_2\bar{\Psi}_2\bar{\sigma}(12\alpha^2 + 10\pi\alpha - 21\bar{\tau}\pi - 18\alpha\bar{\tau} + 6\pi^2$$
$$+44\bar{\Psi}_2 - 12\alpha\bar{\alpha} - 21\tau\pi - 18\alpha\tau - 14\pi\bar{\alpha}) = 0$$

$$
\begin{aligned}
\Psi_2 \quad = \quad & -\tfrac{72}{11}\bar{\pi}\bar{\alpha} + \tfrac{45}{11}\bar{\tau}^2 + \tfrac{4}{11}\Phi_{11} - \tfrac{78}{11}\bar{\alpha}\bar{\tau} - \tfrac{108}{11}\bar{\pi}\bar{\tau} + \tfrac{90}{11}\tau\pi \\
& -\tfrac{10}{11}\pi\bar{\pi} - \tfrac{16}{11}\Lambda - \tfrac{15}{22}\pi^2 - \tfrac{12}{11}\alpha\bar{\alpha} + \tfrac{105}{11}\bar{\tau}\pi + \tfrac{60}{11}\pi\bar{\alpha} \\
& -\tfrac{9}{11}\tau\bar{\tau} - \tfrac{54}{11}\tau^2 - \tfrac{180}{11}\tau\bar{\alpha} - \tfrac{108}{11}\bar{\alpha}^2 + \tfrac{90}{11}\alpha^2 + \tfrac{9}{11}\bar{\pi}^2 \\
& +\tfrac{54}{11}\alpha\tau - \tfrac{72}{11}\alpha\bar{\pi} + \tfrac{60}{11}\pi\alpha - \tfrac{126}{11}\bar{\pi}\tau + \tfrac{150}{11}\alpha\bar{\tau}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
0 &= \pi + 2\alpha \\
0 &= \bar{\pi} + 2\bar{\alpha} \\
0 &= 2\alpha + 3\bar{\tau} + 2\bar{\alpha} + 3\tau \\
0 &= 27\bar{\tau}^2 + 12\bar{\alpha}\bar{\tau} + 4\alpha^2 - 40\alpha\bar{\alpha} + 24\alpha\bar{\tau} \\
\\
0 &= -2188\bar{\alpha}\alpha\bar{\tau} + 6294\tau\alpha\bar{\tau} - 1584\tau\bar{\alpha}^2 \\
&\quad +1188\tau^2\bar{\alpha} - 7172\bar{\tau}\bar{\alpha}^2 - 5048\bar{\alpha}^2\alpha \\
&\quad +2824\alpha^2\bar{\tau} + 3465\bar{\tau}\tau^2 + 1278\alpha\bar{\tau}^2 \\
&\quad -1584\bar{\alpha}^3 - 1984\tau\bar{\alpha}\alpha - 3960\tau\bar{\alpha}\bar{\tau} \\
&\quad +2456\tau\alpha^2 + 2310\alpha\tau^2 - 1386\bar{\tau}^3 \\
&\quad +904\alpha^3 + 2277\tau\bar{\tau}^2 - 5742\bar{\alpha}\bar{\tau}^2 \\
&\quad +608\bar{\alpha}\alpha^2 \\
\\
0 &= 1396\bar{\alpha}\alpha\bar{\tau} + 1302\tau\alpha\bar{\tau} - 396\bar{\tau}\bar{\alpha}^2 \\
&\quad +1320\bar{\alpha}^2\alpha + 2248\alpha^2\bar{\tau} + 297\bar{\tau}\tau^2 \\
&\quad +4734\alpha\bar{\tau}^2 + 1320\tau\bar{\alpha}\alpha - 396\tau\bar{\alpha}\bar{\tau} \\
&\quad -3240\tau\alpha^2 - 990\alpha\tau^2 - 198\bar{\tau}^3 \\
&\quad -664\alpha^3 - 99\tau\bar{\tau}^2 - 66\bar{\alpha}\bar{\tau}^2 \\
&\quad -2512\bar{\alpha}\alpha^2
\end{aligned}
}
$$

$$
\boxed{\bar{\alpha} = \tau = 2\alpha + 3\bar{\tau} = 0}
$$