Harmonic Maps and the Eells-Sampson Theorem

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Abstract

The definition and examples of harmonic maps between Riemannian manifolds will be given. The Eells-Sampson Theorem will be stated and its proof outlined.

The Eells-Sampson states that any smooth map from a compact Riemannian manifold into a compact Riemannian manifold with non-positive sectional curvature is homotopic to a harmonic map.

What Is a Harmonic Map?

Recall geodesics: Let $\gamma : [a,b] \longrightarrow M$ be a smooth curve in a Riemannian manifold M.

The *arclength* of γ is defined to be:

$$L(\gamma) := \int_a^b |\dot{\gamma}(t)| \, dt.$$

The *energy* of γ is defined to be:

$$E(\gamma) := \frac{1}{2} \int_a^b |\dot{\gamma}(t)|^2 \, dt.$$

Fact: Minimizing the energy with respect to fixed endpoints yields a curve which is locally arclength-minimizing and parametrized by arclength.

A Necessary Condition for a Minimizer: The Euler Equation

Let $\gamma := [0,1] \longrightarrow M$ be a smooth curve. Let $F : [0,1] \times (-\epsilon, \epsilon) \longrightarrow M$ be a smooth map such that $\gamma(t) = F(t,0)$, $F(0,s) = \gamma(0)$ and $F(1,s) = \gamma(1)$. Such an $\{F(\cdot,s)\}_s$ is called a smooth variation of γ with fixed end points.

Then, computations show:

$$\left.\frac{d}{ds}\right|_{s=0} E(F(\cdot,s)) = -\int_0^1 \left\langle \left.\frac{\partial F}{\partial s}\right|_{s=0}, \nabla_{\dot\gamma}\dot\gamma \right\rangle dt$$

If γ is to minimize energy, we must have $\left.\frac{d}{ds}\right|_{s=0} E(F(\cdot,s)) = 0$, for any such smooth variation $F(\cdot,s)$ of γ. Hence

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

is a necessary condition. This is known as the Euler equation for the energy functional.

The Euler equation for Energy in Local Coordinates

It looks like:

$$\ddot{x}^i + \Gamma^i_{jk}(x(t))\dot{x}^j\dot{x}^k = 0,$$

where $x^i(t), i = 1, \ldots, m := \dim M$ are the local coordinate parametrization of γ.

A smooth curve γ is called a *geodesic* if it satisfies Euler equation

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

We remark that this is also the equation for autoparallelness.

Generalization: Domain being a Riemannian manifold

Let $u : (M, g) \longrightarrow (N, h)$ be a map between Riemmannian manifolds with M compact (for this motivation only).

Mimicking the geodesic case, we want to say that u is harmonic if u satisfies the Euler equation associated to minimizing:

$$E(u) := \int_M \frac{1}{2} |du|^2 \, d\mu_g$$

with respect to "near-by" smooth maps (with fixed boundary values? Forget this for now — we assume M and N are boundaryless).

Got a problem: What is the norm $|du|$ of the differential du?

Naturally Induced Norm for du

Oberseve that:

$$du \in \Gamma(T^*M \otimes u^{-1}TN)$$

Fact: The metrics on (M, g) and (N, h) naturally induce a metric on the vector bundle $T^*M \otimes u^{-1}TN$ via:

$$\left\langle dx^i \otimes \frac{\partial}{\partial y^\alpha} \circ u, dx^j \otimes \frac{\partial}{\partial y^\beta} \circ u \right\rangle = g^{ij}(x)\, h_{\alpha\beta}(u(x))$$

In coordinates,

$$|du_x|_x^2 = g^{ij}(x) h_{\alpha\beta}(u(x)) \frac{\partial u^\alpha}{\partial x^i}(x) \frac{\partial u^\beta}{\partial x^j}(x)$$

$|du|^2$ Measures the Amount of Local Stretching by $u : M \longrightarrow N$

Let $n := \dim N$ and $m := \dim M$. We may represent the linear map $du_x : T_xM \longrightarrow T_{u(x)}N$ by an $n \times m$ matrix $(\lambda_i^\alpha)_{i=1\ldots m,\alpha=1\ldots n}$, where

$$du(e_i) = \lambda_i^\alpha e'_\alpha,$$

and e_i, e'_α are orthonormal bases for T_xM and $T_{u(x)}N$ respectively.

Then

$$\begin{aligned} |du_x|^2 &= \sum_{i=1}^{m} \sum_{\alpha=1}^{n} (\lambda_i^\alpha)^2 \\ &= \sum_{i=1}^{m} |du(e_i)|^2_{T_{u(x)}N} \end{aligned}$$

Defintion of Harmonic Maps

Now the energy

$$E(u) := \int_M \frac{1}{2} |du|^2 \, d\mu_g$$

of a map $u : M \longrightarrow N$ is well-defined whenever M and N are Riemannian with M compact.*

The associated Euler equations, in local coordinates, are:

$$\Delta u^\alpha + g^{ij} \Gamma'^{\alpha}_{\beta\gamma}(u) \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} = 0,$$

where $\Delta u^\alpha = g^{ij} \left\{ \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial u^\alpha}{\partial x^k} \right\}$, the Laplacian of u^α on M.

$u : (M, g) \longrightarrow (N, h)$ is said to be *harmonic* if it satisfies the above equations.

*Note that the definition of harmonicity requires no compactness of M. It is $E(u)$ that requires it, but $E(u)$ here serves only a motivational purpose.

Another Way of Looking at the Harmonic Map Equation

Recall that $u : (M, g) \longrightarrow (N, h)$ and $du \in \Gamma(T^*M \otimes u^{-1}TN)$. The Levi-Civita connections on M and N naturally induce a connection on $\Gamma(T^*M \otimes u^{-1}TN)$. For du, it can be given by

$$\nabla du(X, Y) = \overset{N}{\nabla}_{du(X)} du(Y) - du\left(\overset{M}{\nabla}_X Y\right).$$

Note that $\nabla du \in \Gamma(T^*M \otimes T^*M \otimes u^{-1}TN)$ is a “3-index thing” — two from M and one from N.

We can take the trace of ∇du (over its two indices from M of course) by

$$\tau(u) := (\mathrm{trace}\, \nabla du)(x) = \sum_{i=1}^{m} (\nabla du)_x(e_i, e_i),$$

where the e_i form a g-orthonormal basis for T_xM.

$\tau(u)$, known as the *tension field* of u, is thus a vector field on M with values in the tangent spaces of N, i.e. $\tau(u) \in \Gamma(u^{-1}TN)$.

Another Way of Looking at the Harmonic Map Equation (cont'd)

In local coordinates, the expression for $\tau(u)$ is exactly the LHS of the harmonic map equation, which can therefore be rewritten as:

$$\tau(u) := \text{trace}\, \nabla du = 0.$$

It is in this sense that a map $u : M \longrightarrow N$ is harmonic if its the trace of its Hessian (second derivative) vanishes identically.

We stress again that the definition of harmonicity requires no compactness of either manifold.

Examples of Harmonic Maps

$$\begin{aligned} 0 &= \tau(u)^\alpha \\ &= \Delta u^\alpha + g^{ij}\Gamma'^{\alpha}_{\beta\gamma}(u)\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j} \\ &= g^{ij}\left\{\frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial u^\alpha}{\partial x^k} + \Gamma'^{\alpha}_{\beta\gamma}(u)\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j}\right\} \end{aligned}$$

- Constant maps (absolute minimizers of energy when M is compact.)
- Identity maps ($du_x = \mathrm{id}_{T_xM}$ and $\nabla du = 0$)
- Harmonic functions ($N = \mathbb{R}$ so $\Gamma'^{\alpha}_{\beta\gamma} = 0$.)
- Geodesics ($M = S^1$ or $\mathbb{R}$, so $g^{ij} = 1$, $x^i = t$ and $\Gamma^i_{jk} = 0$.)

Examples of Harmonic Maps (cont'd)

Minimal Submanifolds

Suppose $u : M \longrightarrow N$ is an isometric immersion; in other words, M is an immersed Riemannian submanifold of N. Then the second fundamental form (or shape tensor) of M is defined by:

$$II(u)(X,Y) := \overset{N}{\nabla}_{du(X)} du(Y) - du\left(\overset{M}{\nabla}_X Y\right),$$

which in fact coincides with ∇du.

On the other hand, the *mean curvature vector field* of the Riemannian submanifold M is defined to be $H := \frac{1}{m}\text{trace}\, II(u)$, and M is called a *minimal submanifold* of N if $H = 0$.

Hence an isometric immersion $u : M \longrightarrow N$ is harmonic if and only if M is a minimal submanifold of N.

The Eells-Sampson Theorem

(An Existenc Result for Harmonic Maps)

Suppose M and N are compact Riemannian manifolds with N of non-positive sectional curvature. Then every smooth map $f \in C^{\infty}(M, N)$ is homotopic to a harmonic map from M into N.

Idea of Proof (The Heat Flow Method)

$E(u) := \int_M \frac{1}{2}|du|^2\, d\mu_g \geq 0$ is a (directionally) differentiable functional on $C^\infty(M,N)$, a Banach manifold, and a computation shows:

$$-\mathrm{grad}E(u) = \tau(u).$$

So starting from $f \in C^\infty(M,N)$, we attempt to "move along" $C^\infty(M,N)$ in the direction of "steepest descent" in E.[†] We thus seek a solution $u : M \times [0,T) \longrightarrow M$ to

$$\left\{ \begin{array}{rcl} \frac{\partial u}{\partial t}(x,t) & = & \tau(u)(x,t), \\ u(x,0) & = & f(x), \end{array} \right. \tag{1}$$

an IVP with a semilinear parabolic system of PDEs on M.

[†] K. Uhlenbeck has indeed proved the Eells-Sampson theorem using infinite-dimensional Morse theory.

Idea of Proof (cont'd)

Under the compactness and curvature assumptions, this IVP has a unique solution on $M \times [0, \infty)$ such that $u(\cdot, t)$ converges uniformly to a harmonic map $u_\infty \in C^{2+\alpha}(M, N)$ as $t \to \infty$, i.e. $\sup_{x \in M} d(u(x,t), u_\infty(x)) \to 0$, as $t \to \infty$.

The harmonic map equation $\tau(u) = 0$ is a semilinear elliptic system of PDEs on M. Standard "elliptic bootstrapping" shows that any $C^{2+\alpha}(M, N)$ solution of the harmonic map equation is in fact in $C^\infty(M, N)$.

Also, by "parabolic bootstrapping," a $C^{2+\alpha,1+\alpha}$ solution of (1) is in fact $C^{\infty,\infty}$.

Outline of Proof

Define

$$\begin{array}{rcll} u_t & := & u(\cdot,t) & \\ e(u_t) & := & \frac{1}{2}|du_t|^2 & \text{energy density} \\ E(u_t) & := & \int_M e(u_t)\,d\mu_g & \text{energy} \\ \kappa(u_t) & := & \frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 & \text{kinetic energy density} \\ K(u_t) & := & \int_M \kappa(u_t)\,d\mu_g & \text{kinetic energy} \end{array}$$

Then computations show:

$$\begin{aligned} \frac{d}{dt}E(u_t) &= -2K(u_t) \leq 0, \\ \frac{d^2}{dt^2}E(u_t) &= -2\frac{d}{dt}K(u_t). \end{aligned}$$

Weitzenböck Formulae

If $u : M \times [0, T) \longrightarrow M$ solves (1), then

$$\begin{aligned}
\frac{\partial e(u_t)}{\partial t} &= \Delta e(u_t) - |\nabla\nabla u_t|^2 \\
&\quad - \sum_{i=1}^{m} \left\langle du_t \left(\sum_{j=1}^{m} \overset{M}{\mathrm{Ric}}(e_i, e_j) e_j \right), du_t(e_i) \right\rangle \\
&\quad + \sum_{i,j=1}^{m} \left\langle \overset{N}{R}_{du_t(e_i), du_t(e_j)} du_t(e_j), du_t(e_i) \right\rangle, \text{ and} \\
\frac{\partial \kappa(u_t)}{\partial t} &= \Delta \kappa(u_t) - \left| \nabla \frac{\partial u_t}{\partial t} \right|^2 + \sum_{i=1}^{m} \left\langle \overset{N}{R}_{du_t(e_i), \frac{\partial u_t}{\partial t}} \frac{\partial u_t}{\partial t}, du_t(e_i) \right\rangle
\end{aligned}$$

Weitzenböck Formulae (cont'd)

$$\frac{\partial \kappa(u_t)}{\partial t} = \Delta \kappa(u_t) - \left| \nabla \frac{\partial u_t}{\partial t} \right|^2 + \sum_{i=1}^{m} \left\langle \overset{N}{R}_{du_t(e_i), \frac{\partial u_t}{\partial t}} \frac{\partial u_t}{\partial t}, du_t(e_i) \right\rangle$$

Hence if N has non-positive sectional curvature, then

$$\frac{d^2}{dt^2} E(u_t) = -2 \frac{d}{dt} K(u_t) \geq 0,$$

since

$$\begin{aligned} \frac{d}{dt} K(u_t) &= \frac{d}{dt} \int_M \kappa(u_t)\, d\mu_g \\ &= \int_M \frac{\partial}{\partial t} \kappa(u_t)\, d\mu_g \\ &\leq \int_M \Delta \kappa(u_t)\, d\mu_g = 0. \end{aligned}$$

Then for any $C^{2+\alpha,1+\alpha}$ solution $u : M \times [0,T) \longrightarrow M$ of (1), $E(u_t)$ is non-increasing and if N has non-positive sectional curvature, then $\frac{d}{dt}E(u_t) = -2K(u_t)$ is non-decreasing.

Claim 1: The IVP (1) has a (temporally) global solution u_t on all of $M \times [0,\infty)$.

Then, $E(u_t)$ must plateau as $t \to \infty$, i.e. $K(u_t) := \int_M \frac{1}{2}|\frac{\partial}{\partial t}u_t|^2 \, d\mu_g \to 0$ as $t \to \infty$. Hence $\frac{\partial}{\partial t}u_t \to 0$ μ_g-almost everywhere on M as $t \to \infty$.

Claim 2: All (temporal and spatial) derivatives of u_t up to second order converge uniformly on M as $t \to \infty$.

Say, $u_t \to u_\infty$. Then

$$\begin{array}{ccc} \frac{\partial}{\partial t}u_t & = & \tau(u_t) \\ \downarrow & & \downarrow \\ 0 & & \tau(u_\infty) \end{array}$$

Hence $\tau(u_\infty) = 0$, i.e. u_∞ is a $C^{2+\alpha}$ harmonic map hence a C^∞ one.

Since the convergence is uniform, by considering a finite cover of M by geodesically convex domains, we see that $f = u_0$ and u_∞ are homotopic.

Now the Fine Print!

We want to use functional analysis, thus need Banach spaces. Hence, use Nash's isometric imbedding theorem to embed N into some Euclidean space $\mathbb{R}^q$.

By compactness of N, there exists a tubular neighbourhood with projection $\pi : \tilde{N} \longrightarrow N$. Extend the RHS of the heat equation to $u \in C^{2+\alpha,1+\alpha}(M, \tilde{N})$ in the obvious way using π — namely, replace the Christoffel's symbols of N that appear in $\tau(\cdot)$ with the Hessian of π; they agree on N.

Extend again to all of $\mathbb{R}^q$ by smoothing it to zero near $\partial\tilde{N}$. We can now consider the same problem but for $u \in C^{2+\alpha,1+\alpha}(M, \mathbb{R}^q)$, now a function space.

We need to ensure that the solutions to the extended problem coincide with those of the original problem. This can be argued as follows:

If a solution $\tilde{u}$ to the extended problem has values in $\tilde{N}$, then a computation shows that $|\tilde{u} - \pi(\tilde{u})|^2 \geq 0$ achieves its maximum on $M \times \{0\}$. (Curvature assumption on N used here.)

But since the initial value f has values in N, this difference is identically zero on $M \times [0, T)$, hence $\tilde{u}$ in fact has values in N and solves our original heat flow problem.

Now every solution $u \in C^{2+\alpha,1+\alpha}(M \times [0, T), \mathbb{R}^q)$ initially has values in $\tilde{N}$ since the initial value f has values in N.

Proof of Claim 1

- **Short-time existence:** Consider $P(u_t) = \frac{\partial u_t}{\partial t} - \tau(u_t)$ as a map between suitably chosen Banach spaces. Linearizing P yields a linear parabolic IVP, and standard linear parabolic theory implies this linear IVP has unique solution, with the solution operator being a bounded map. Existence/uniqueness of solution translates to the solution operator being a linear isomorphism. The Open Mapping Theorem for Banach spaces implies it is a Banach space isomorphism. The Inverse Function Theorem for Banach spaces then implies the original nonlinear map P admits a local inverse. This translates to existence of short-time solution.

- **A priori bounds:** Let $u \in C^{2,1}(M \times [0,T), \mathbb{R}^q) \cap C^\infty(M \times (0,T), \mathbb{R}^q)$ solve the extended IVP of (1). Then for any $0 < \alpha < 1$, there exists a constant $C = C(M, N, f, \alpha) > 0$ such that

  $$|u_t|_{C^{2+\alpha}(M,\mathbb{R}^q)} + \left|\frac{\partial u_t}{\partial t}\right|_{C^\alpha(M,\mathbb{R}^q)} \le C,$$

  for all $t \in [0,T)$. Now,

  $$\begin{aligned} |v|_{C^\alpha(M,\mathbb{R}^q)} &= |v|_{C^{0+\alpha}(M,\mathbb{R}^q)} \\ := \; & \|v\|_{\infty,M} + \sup_{\substack{x,y \in M \\ x \neq y}} \frac{|v(x) - v(y)|}{d(x,y)^\alpha}, \end{aligned}$$

  and the norms on $C^{k+\alpha}(M,\mathbb{R}^q)$, $C^{k+\alpha,l+\alpha}(M \times [0,T), \mathbb{R}^q)$, etc. are similarly defined (but with control on all spatial derivatives of order up to k and temporal derivative of order up to l.) Similar a priori bounds hold for solutions to (1).

- Note that even though the norms above seem convoluted, their a priori bounds precisely say that the family of functions $\{u(\cdot,t)\}_{t\geq 0}$, indexed by t, as well as all of its (temporal and spatial) derivatives are $\|\cdot\|_{\infty,M}$-bounded in $C^0(M,\mathbb{R}^q)$ and these families are all equicontinuous (thanks to the control on the difference quotients)!

- **Long-time existence:**

$$T := \sup\{t \in [0,\infty) \,|\, \text{Extended(1) has a solution on } [0,t)\}.$$

Suppose, on the contrary, that T is finite.

Let $t_i \to T$. The a priori bounds imply $\{u(\cdot,t_i)\}_i$ as well as the corresponding sequences of its derivates are all uniformly bounded on M and equicontinuous.

(Repeated use of) the Ascoli-Arzela theorem implies there is a subsequence of $\{t_i\}$ such that $\{u(\cdot,t_i)\}_i$ and each of its derivatives up to second order converge uniformly and the limit functions satisfy the PDEs involved.

Uniform bound on $\frac{\partial u_t}{\partial t}$ furthermore implies we arrive at unique limit functions regardless of how the t_i approach T.

All of this implies we can extend the original flow from $M \times [0,T)$ to $M \times [0,T]$. But we now can invoke short-time existence again at $t = T$ and extend the flow beyond T, contradicting the definition of T. This proves $T = \infty$.

Proof of Claim 2

(*All derivatives of* $u(x, t)$ *up to second order converge uniformly on* M *as* $t \to \infty$*.*)

Automatic by choice of the function spaces above!

******** *QED* ********