On the Geometry of the Moduli Space of Real Binary Octics

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The Moduli Space of Stable Real Octics

Intention: To produce a non-arithmetic lattice in $\text{Isom}(\mathbb{RH}^5)$.

$$\mathcal{M}_s^{\mathbb{R},\text{cubic surfaces}} \cong \bigcup_{i=1}^{5} \underbrace{\Gamma_i^{\mathbb{R},\text{cs}} \backslash \mathbb{RH}^4}_{\mathcal{M}_i^{\mathbb{R},\text{cs}}} \cong \Gamma^{\mathbb{R},\text{cs}} \backslash \mathbb{RH}^4.$$

Hope same phenomenon will occur with $\mathcal{M}_s^{\mathbb{R},\text{binary octics}}$, i.e.

$$\mathcal{M}_s^{\mathbb{R}} \overset{?}{\cong} \bigcup_{i=0}^{4} \underbrace{\Gamma_i^{\mathbb{R}} \backslash \mathbb{RH}^5}_{\mathcal{M}_{s,i}^{\mathbb{R}}} \overset{?}{\cong} \Gamma^{\mathbb{R}} \backslash \mathbb{RH}^5.$$

Summary of New Results

$$\mathcal{M}_s^{\mathbb{R}} \overset{?}{\cong} \underbrace{\bigcup_{i=0}^{4} \Gamma_i^{\mathbb{R}} \backslash \mathbb{RH}^5}_{\mathcal{M}_{s,i}^{\mathbb{R}}} \overset{?}{\cong} \Gamma^{\mathbb{R}} \backslash \mathbb{RH}^5.$$

i	0	1	2	3	4
# real points	8	6	4	2	0
# cx. conj. pairs	0	1	2	3	4

1. $\Gamma_0^{\mathbb{R}}, \ldots, \Gamma_3^{\mathbb{R}}$ have been found explicitly. Each is a finite-index subgroup of an arithmetic reflection subgroup $\mathbb{P}A_i \subset \mathbb{P}\text{Isom}(\mathbb{RH}^5)$. The Coxeter diagrams of $A_0, \ldots, A_3$ have been worked out.
2. $\Gamma_4^{\mathbb{R}}$ is narrowed down to one of two possibilities. One of these possibilities is shown to be a finite-index subgroup of an arithmetic subgroup in $\mathbb{P}\text{Isom}(\mathbb{RH}^5)$ and its Coxeter diagram has been worked out.
3. (The A-C-T construction of) $\mathcal{M}_s^{\mathbb{R}}$ cannot admit a real hyperbolic orbifold structure: Points in the stratum $\Delta^{0,1}$ are not locally real hyperbolic (modulo a finite group of isometries).

Strategy

1. Start with

$$\begin{tikzcd} \mathcal{M}_s \arrow[d, equal] & \cong & \mathbb{P}\Gamma\backslash\mathbb{C}\mathbb{H}^5 \\ \left(\mathbb{C}\mathbb{P}^8 - \Delta_{\geq 4}\right)/\mathrm{PGL}(2,\mathbb{C}) & & \end{tikzcd}$$

[Deligne-Mostow]

2. A-C-T observed: Periods (in $\mathbb{C}\mathbb{H}^5$) of real octics lie in $\bigcup_{\chi\in ?} \mathbb{R}\mathbb{H}^5_\chi \subset \mathbb{C}\mathbb{H}^5$.

   So,

$$\begin{tikzcd} \mathcal{M}_s^{\mathbb{R}} \arrow[d, equal] & \cong & \mathbb{P}\Gamma \Big\backslash \left\{ \left(\coprod_{\chi\in ?} \mathbb{R}\mathbb{H}^5_\chi \right) \Big/ \approx \right\} =: \mathbb{P}\Gamma\backslash\mathcal{K}_s \\ \left(\mathbb{R}\mathbb{P}^8 - \Delta^{\mathbb{R}}_{\geq 4}\right)/\mathrm{PGL}(2,\mathbb{R}) & & \end{tikzcd}$$

   $\coprod$ is to undo $\mathrm{PGL}(2,\mathbb{C})$-quotienting. $\approx$ is to impose $\mathrm{PGL}(2,\mathbb{R})$-quotienting.

3. Study the geometry of $\mathbb{P}\Gamma\backslash\mathcal{K}_s =:$ A-C-T construction of the moduli space of stable real binary octics.

Deligne-Mostow Construction of $\mathcal{M}_s$

Let $p(x) \in \mathcal{P}_0$. Define X_p to be the completion of

$$\left\{ \; (x,y) \in \mathbb{C}^2 \;\middle|\; y^4 - p(x) = 0 \; \right\} =: X_p \xrightarrow{\sigma} X_p : (x,y) \mapsto (x, \sqrt{-1}y).$$

X_p is a Riemann surface and the map $X_p \xrightarrow{\pi} \mathbb{C} : (x,y) \mapsto x$ is a quadruple cyclic covering of $\mathbb{CP}^1$ branched over the 8 distinct roots of $p(x)$. Each ramification point has ramification index 4. Riemann-Hurwitz $\Rightarrow$ genus(X_p) $= H^{1,0}(X_p) =$ 9 and $\mathrm{rank}_{\mathbb{Z}} H^1(X_p, \mathbb{Z})$ = $\dim_{\mathbb{C}} H^1(X_p, \mathbb{C}) = 2 \cdot 9 = 18$.

Let $\Lambda(X_p) := H^1_{\sigma^2=-1}(X_p, \mathbb{Z})$. Then, $\Lambda(X_p) \xrightarrow{\sigma^*} \Lambda(X_p)$ satisfies $\sigma^2 + 1 = 0$. Hence $\Lambda(X_p)$ becomes a $\mathbb{Z}[\sqrt{-1}]$-module via

$$-\sqrt{-1} \cdot \phi := \sigma^*(\phi).$$

FACT: $\Lambda(X_p) \cong \mathbb{Z}[\sqrt{-1}]^6$.

Hermitian Form on $\Lambda(X_p)$

Consider the embedding

$$\begin{tikzcd}
\Lambda(X_p) \arrow[r, hook] \arrow[d, equal] & H^1_{\sigma^2=-1}(X_p,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[r, phantom, "\cong"] & H^1_{\sigma^2=-1}(X_p,\mathbb{C}) \arrow[d, equal] \\
H^1_{\sigma^2=-1}(X_p,\mathbb{Z}) & & H^1_{\sigma=-\sqrt{-1}}(X_p,\mathbb{C}) \oplus H^1_{\sigma=\sqrt{-1}}(X_p,\mathbb{C}) \arrow[d] \\
& & H^1_{\sigma=-\sqrt{-1}}(X_p,\mathbb{C})
\end{tikzcd}$$

Computations show

$$\left(\begin{array}{c} \underbrace{H^1_{\sigma=-\sqrt{-1}}(X_p,\mathbb{C})} \\ \| \\ \underset{(+)}{H^{1,0}_{\sigma=-\sqrt{-1}}(X)} \oplus \underset{(-----)}{H^{0,1}_{\sigma=-\sqrt{-1}}(X)} \end{array} , \quad (\alpha,\beta) \xmapsto{h'} 2\sqrt{-1}\int_{X_p} \alpha \wedge \overline{\beta} \right) \cong \mathbb{C}^{1,5}$$

Hermitian Form on $\Lambda(X_p)$ (cont'd)

$$\Lambda(X_p) \;=\; H^1_{\sigma^2=-1}(X_p,\mathbb{Z}) \;\hookrightarrow\; \Big(H^1_{\sigma=-\sqrt{-1}}(X_p,\mathbb{C})\,,\,h'\Big) \;\cong\; \mathbb{C}^{1,5}$$

The pullback Hermitian form on $\Lambda(X_p)$ is given by:

$$h(\xi,\eta) = -\,\Omega(\xi,\sigma(\eta)) - \sqrt{-1}\,\Omega(\xi,\eta),$$

where $\Omega(\xi,\eta) := \langle\, \xi \cup \eta\,,\; X_p\,\rangle$.

Computations show $(\Lambda(X_p)\,,\,h)$ is abstractly isometric to $\Lambda := \mathbb{Z}[\sqrt{-1}]^6$, equipped with

$$\begin{bmatrix} -2 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & -2 \end{bmatrix} \oplus \begin{bmatrix} -2 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & -2 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & 0 \end{bmatrix}$$

Explanation of $\begin{bmatrix} -2 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & -2 \end{bmatrix}$

The vanishing cohomology corresponding to a nodal octic has $\mathbb{Z}[\sqrt{-1}]$-rank one and is generated by a vector of norm -2.

The vanishing cohomology corresponding to a cuspidal octic is an orthogonal summand of $\Lambda(X_p) \cong \Lambda$ of $\mathbb{Z}[\sqrt{-1}]$-rank two with inherited $\mathbb{Z}[\sqrt{-1}]$-Hermitian form:

$$\left(\mathbb{Z}[\sqrt{-1}]^2 \, , \, \begin{bmatrix} -2 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & -2 \end{bmatrix} \right)$$

OUTLINE OF PROOF

Local “pictorial” computations of the intersection form of the vanishing 1-homology of $H_{1,\sigma^2=-1}(X_p,\mathbb{Z})$ over a coalescing two-point or three-point configurations, i.e. the intersection form of the vanishing 1-homology of

$$y^4 = x^2 - \epsilon^2 \quad \text{and} \quad y^4 = x^3 - \epsilon^3, \quad \text{as } \epsilon \longrightarrow 0, \epsilon \geq 0.$$

“Fiberwise” Summary

$$p \in \mathcal{P}_0 \quad \rightsquigarrow \quad \left\{ \begin{array}{c} \text{quadruple cyclic covering } X_p \\ \text{of } \mathbb{CP}^1 \text{ branched at roots of } p \\ \text{with cyclic action } X_p \xrightarrow{\sigma} X_p \text{ of order 4} \end{array} \right.$$

$$\rightsquigarrow \quad \mathbb{Z}[\sqrt{-1}]\text{-lattice } \Lambda(X_p) := \Big(H^1_{\sigma^2=-1}(X,\mathbb{Z})\,,\ h \Big) \cong \Lambda$$

Recall $\Lambda \otimes_{\mathbb{Z}[\sqrt{-1}]} \mathbb{C} \cong \mathbb{C}^{1,5} = \mathbb{C}^{1+,5-}$, and $\Lambda := \mathbb{Z}[\sqrt{-1}]^6$ equipped with

$$\begin{bmatrix} -2 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & -2 \end{bmatrix} \oplus \begin{bmatrix} -2 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & -2 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & 0 \end{bmatrix}$$

Recall also $H^{1,0}_{\sigma=-\sqrt{-1}}(X_p)$ is a positive 1-dimensional subspace of

$$\Big(H^1_{\sigma=-\sqrt{-1}}(X_p,\mathbb{C})\,,\ h \Big) \cong \mathbb{C}^{1,5} = \mathbb{C}^{1+,5-}.$$

Hence, $H^{1,0}_{\sigma=-\sqrt{-1}}(X_p) \in \mathbb{CH}\Big(H^1_{\sigma=-\sqrt{-1}}(X_p,\mathbb{C}) \Big) \cong \mathbb{CH}^5$

Construction of $\mathcal{F}_0$ = Domain(Period Map) $\longrightarrow \mathcal{P}_0$

A *framed smooth form* over $p \in \mathcal{P}_0$ is a "projective equivalence class" of an (abstract) isometry of $\Lambda(X_p) \xrightarrow{\sim} \Lambda$, where two such isometries are "projectively equivalent" if one is a $\mathbb{Z}[\sqrt{-1}]$-unit scalar multiple of the other.

$\mathcal{F}_0$ is the space of all framed smooth forms, and we get a natural projection map $\mathcal{F}_0 \to \mathcal{P}_0$, which is in fact an unbranched covering.

GL$(2, \mathbb{C})$-action on $\mathcal{P}_0$ naturally extends to $\mathcal{F}_0$ via induced action on cohomology. Let $G :=$ GL$(2, \mathbb{C})/\langle \pm 1, \pm\sqrt{-1}\rangle$. Then G acts freely on $\mathcal{F}_0$.

FACTS

1. Let $\mathbb{P}\Gamma$ be the Deck transformation group of the covering $\mathcal{F}_0 \to \mathcal{P}_0$. Then we know $\mathbb{P}\Gamma \subseteq \mathbb{P}$Isom$(\Lambda)$, since for each $p \in \mathcal{P}_0$, $\Lambda(X_p)$ is abstractly isometric to Λ and each Deck transformation must preserve $\Lambda(X_p)$. Clearly, $\mathbb{P}\Gamma \backslash \mathcal{F}_0 \cong \mathcal{P}_0$.
2. $\mathcal{F}_0$ is also the covering corresponding to the kernel of the representation $\pi_1(\mathcal{P}_0, p_0) \xrightarrow{\rho} \mathbb{P}$Isom$(\Lambda(X_{p_0}))$, where $p_0 \in \mathcal{P}_0$ is some fixed smooth octic. $\mathbb{P}\Gamma$ is thus also the monodromy group of the representation ρ.
3. In fact, $\mathbb{P}\Gamma = \mathbb{P}$Isom$(\Lambda)$.

The Period Map $\mathcal{F}_0 \to \mathbb{CH}^5$

$$\begin{array}{ccc} \mathcal{F}_0 & \longrightarrow & \mathbb{CH}^5 = \mathbb{CH}\left(\Lambda \otimes_{\mathbb{Z}[\sqrt{-1}]} \mathbb{C}\right) \\ \left[\Lambda(X_p) \xrightarrow{f} \Lambda\right] & \longmapsto & f(H^{1,0}_{\sigma=-\sqrt{-1}}(X_p)) \end{array}$$

FACTS

1. The period map is equivariant with respect to the actions of $\mathbb{P}\Gamma$ on $\mathcal{F}_0$ (via Deck transformations ↔ change of basis of projectivized frames) and on $\mathbb{CH}^5$ (via isometries). It is also $(G \curvearrowright \mathcal{F}_0)$-invariant.
2. Let $\mathcal{H} := \bigcup_{r\in\mathcal{R}} \mathbb{CH}(r^\perp) \subset \mathbb{CH}^5$, where $\mathcal{R}$ is the set of all vectors in Λ of norm -2. FACT: the period map maps $\mathcal{F}_0$ onto $\mathbb{CH}^5 - \mathcal{H}$ and it maps $\mathcal{F}_0/G$ biholomorphically onto $\mathbb{CH}^5 - \mathcal{H}$.
3. We now have

$$\mathcal{M}_0 \leftrightarrow \mathcal{P}_0/G \cong (\mathbb{P}\Gamma\backslash\mathcal{F}_0)/G \quad \leftrightarrow \quad \mathbb{P}\Gamma\backslash(\mathcal{F}_0/G) \cong \mathbb{P}\Gamma\backslash\left(\mathbb{CH}^5 - \mathcal{H}\right).$$

(Relevant Properties of the) Fox Completion $\mathcal{F}_s \to \mathcal{P}_s$

The Fox completion $\mathcal{F}_s \supset \mathcal{F}_0$ “fills up” the gaps in $\mathcal{F}_0$ in such a way that:

1. the points of $\mathcal{F}_s - \mathcal{F}_0$ “lie above” $\mathcal{P}_s - \mathcal{P}_0$, the stable but non-smooth octics. Elements of $\mathcal{F}_s$ are called *framed stable forms*,
2. the actions $G \curvearrowright \mathcal{F}_0$ and $\mathbb{P}\Gamma \curvearrowright \mathcal{F}_0$ extend to $\mathcal{F}_s$,
3. the period map $\mathcal{F}_0 \to \mathbb{CH}^5$ extends to $\mathcal{F}_s \twoheadrightarrow \mathbb{CH}^5$, holomorphically, equivariantly, and surjectively,
4. $\mathbb{P}\Gamma \backslash \mathcal{F}_s \cong \mathcal{P}_s$ and $\mathcal{F}_s / G \cong \mathbb{CH}^5$.

The Deligne-Mostow Construction of $\mathcal{M}_s$

$$\mathcal{M}_s \leftrightarrow \mathcal{P}_s/G = \left(\mathbb{P}\Gamma \backslash \mathcal{F}_s\right)/G \quad \leftrightarrow \quad \mathbb{P}\Gamma \backslash \left(\mathcal{F}_s/G\right) \cong \mathbb{P}\Gamma \backslash \mathbb{CH}^5$$

$p \in \mathcal{P}_0^{\mathbb{R}} \quad \rightsquigarrow$ **Involutive Anti-isometry κ_p of $\Lambda(X_p)$**

If $p \in \mathcal{P}_0^{\mathbb{R}}$, then complex conjugation $\mathbb{CP}^1 \xrightarrow{\kappa} \mathbb{CP}^1 : x \mapsto \overline{x}$ induces an antiholomorphic involution κ_p on $X_p := \{ \ (x,y) \in \mathbb{C}^2 \mid y^4 - p(x) = 0 \ \}$ via $(x,y) \mapsto (\ \overline{x}\ ,\ \overline{y}\ )$.

κ_p in turn induces an *involutive (antilinear) anti-isometry* (IAAI) κ_p on $\Lambda(X_p)$.

If $\left[\Lambda(X_p) \xrightarrow{f} \Lambda\right]$ is a framed smooth form over $p \in \mathcal{P}_0^{\mathbb{R}}$, then $\chi_{p,f} := f \circ \kappa_p \circ f^{-1} : \Lambda \to \Lambda$ is an IAAI of Λ.

If $[f_1]$ and $[f_2]$ are framed smooth forms over the same $p \in \mathcal{P}_0^{\mathbb{R}}$, then χ_{p,f_1} and χ_{p,f_2} belong to the same Isom(Λ)-conjugacy class of IAAI's of Λ. Thus $[\chi_p] := [\chi_{p,f}]$ is a well-defined Isom(Λ)-conjugacy class, depending only on p, not on $[f]$.

Let $(p_1, [f_1]), (p_2, [f_2])$ be ordered pairs with $p_1, p_2 \in \mathcal{P}_0^{\mathbb{R}}$ and $[f_1]$, $[f_2]$ being framed smooth forms over p_1, p_2 respectively. If p_1, p_2 are of the same topological type, i.e. one can be deformed to the other via smooth real octics, then we can deform $(p_1, [f_1])$ to some $(p_2, [f_2'])$. Noting that IAAI(Λ) is a lattice in IAAI$(\Lambda \otimes \mathbb{C}) =$ IAAI$(\mathbb{C}^{1,5})$, we see

$$p_1, p_2 \in \mathcal{P}_0^{\mathbb{R}} \text{ of same topological type} \quad \Longrightarrow \quad [\chi_{p_1}] := [\chi_{p_1,f_1}] = [\chi_{p_2,f_2'}] = [\chi_{p_2,f_2}] =: [\chi_{p_2}].$$

FACT: Converse holds.

“Real” Octics Have “Real” Periods

$X_p \overset{\kappa_p}{\to} X_p$ also induces an anti-linear involution on $H^1(X_p, \mathbb{C})$ via

$$\begin{array}{ccc} H^1(X_p,\mathbb{C}) & \longrightarrow & H^1(X_p,\mathbb{C}) \\ \phi & \longmapsto & \overline{(\kappa_p)^*(\phi)} \end{array}$$

This involution preserves both Hodge decomposition and the σ-eigenspaces of $H^1(X_p, \mathbb{C})$. It turns out that κ_p restricts to an IAAI on

$$\Lambda(X_p) \otimes_{\mathbb{Z}[\sqrt{-1}]} \mathbb{C} \cong \underbrace{H^1_{\sigma=-\sqrt{-1}}(X_p,\mathbb{C})}_{\mathbb{C}^{1,5}=\mathbb{C}^{1+,5-}} = \underbrace{H^{1,0}_{\sigma=-\sqrt{-1}}(X_p,\mathbb{C})}_{(+)} \oplus \underbrace{H^{0,1}_{\sigma=-\sqrt{-1}}(X_p,\mathbb{C})}_{(-----)},$$

thereby preserving each summands.

Thus, $H^{1,0}_{\sigma=-\sqrt{-1}}(X_p,\mathbb{C}) \in \mathbb{CH}\left(\Lambda(X_p) \otimes \mathbb{C}\right)$ is fixed by $[\kappa_p]$.

Hence, for a given framed smooth form $[\Lambda(X_p) \xrightarrow{f} \Lambda]$ over $p \in \mathcal{P}_0^{\mathbb{R}}$, its period $f(H^{1,0}_{\sigma=-\sqrt{-1}}(X_p,\mathbb{C})) \in \mathbb{CH}^5 = \mathbb{CH}\left(\Lambda \otimes \mathbb{C}\right)$ is fixed by the projective class $[\chi_p] = [f \circ \kappa_p \circ f^{-1}] \in \mathbb{P}\mathrm{IAAI}(\Lambda)$.

We call an element $x \in \mathbb{CH}^5$ a *real period* if $x \in \mathrm{Fix}([\chi_p])$ for some $\chi_p \in \mathrm{IAAI}(\Lambda)$ arising as described above.

Real Periods Lie on Copies of $\mathbb{RH}^5 \subset \mathbb{CH}^5$

1. For each $\chi \in \textsf{IAAI}(\Lambda)$, the metric on Λ restricts to a metric on the $\mathbb{Z}$-module $\textsf{Fix}(\chi) \cong \mathbb{Z}^6$ of signature $(1+, 5-)$. Thus

$$\textsf{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R} \ \cong \ \mathbb{R}^{1,5} = \mathbb{R}^{1+,5-},$$

   and

$$\begin{array}{ccc} \mathbb{RH}\left(\textsf{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}\right) & \cong & \mathbb{RH}^5 \\ \cap & & \cap \\ \mathbb{CH}\left(\Lambda \otimes_{\mathbb{Z}[\sqrt{-1}]} \mathbb{C}\right) & \cong & \mathbb{CH}^5 \end{array}$$

2. Hence, the periods of real octics lie on copies of real hyperbolic space $\mathbb{RH}^5$ within $\mathbb{CH}^5$.

The Allcock-Carlson-Toledo Construction of $\mathcal{M}_s^{\mathbb{R}}$

$$
\begin{aligned}
\mathcal{M}_s^{\mathbb{R}} \quad &:= \quad \mathbb{P}(\mathcal{P}_s^{\mathbb{R}}) \;/\; \mathsf{PGL}(2,\mathbb{R}) \;\leftrightarrow\; \mathcal{P}_s^{\mathbb{R}} \,/\, \mathsf{GL}(2,\mathbb{R}) \\
&\leftrightarrow \quad \mathcal{P}_s^{\mathbb{R}} \;/\; \left(\, \mathsf{GL}(2,\mathbb{R})/\langle \pm 1 \rangle \,\right) \;=:\; \mathcal{P}_s^{\mathbb{R}}/G^{\mathbb{R}} \\
&\leftrightarrow \quad \Big\{ \mathbb{P}\Gamma \setminus \Big(\text{preimage of } \mathcal{P}_s^{\mathbb{R}} \text{ under } \mathcal{F}_s \to \mathcal{P}_s \Big) \Big\} \Big/ \; G^{\mathbb{R}} \\
&=: \quad \Big\{ \mathbb{P}\Gamma \setminus \mathcal{F}_s^{\mathbb{R}} \Big\} \Big/ \; G^{\mathbb{R}} \\
&\leftrightarrow \quad \mathbb{P}\Gamma \;\Big\backslash \Big\{ \mathcal{F}_s^{\mathbb{R}} \,/G^{\mathbb{R}} \Big\} \\
&\cong \quad \mathbb{P}\Gamma \;\left\backslash \left\{ \left(\coprod_{[\chi] \in \mathbb{P}\mathsf{IAAI}^{\mathbb{R}}(\Lambda)} \mathbb{RH}^5_{[\chi]} \right) \Big/ \; \approx \right\} \right. \\
&=: \quad \mathbb{P}\Gamma \setminus \mathcal{K}_s \quad =: \quad \text{A-C-T construction of } \mathcal{M}_s^{\mathbb{R}}
\end{aligned}
$$

Uniformizations of $\mathcal{M}^{\mathbb{R}}_{s,i}$ $(i = 0, \dots, 4)$

Recall: $\mathcal{M}^{\mathbb{R}}_{s} \leftrightarrow \mathbb{P}\Gamma \backslash \mathcal{K}_s = \mathbb{P}\Gamma \Bigg\backslash \left\{ \left(\coprod_{[\chi] \in \mathbb{P}\mathsf{IAAI}^{\mathbb{R}}(\Lambda)} \mathbb{RH}^5_{[\chi]} \right) \Bigg/ \approx \right\}.$

FACTS

1. There are either 6 (or 7) $\mathbb{P}\Gamma = \mathbb{P}\mathsf{Isom}(\Lambda)$-conjugacy classes of IAAI's of Λ. Five of them correspond to $\kappa_{\mathbb{CP}^1}$, and the remaining one to the antipodal map on $\mathbb{CP}^1$.
2. $\mathbb{P}\Gamma$ obviously acts transitively on the collection of the copies $\mathbb{RH}^5_{\chi}$, where all the χ belong to one $\mathbb{P}\Gamma$-conjugacy class; equivalently, the corresponding octics have the same topological type.

It should now be clear that

$$\mathcal{M}^{\mathbb{R}}_{s,i} \cong \mathsf{Stab}_{\mathbb{P}\Gamma}\underbrace{(\mathsf{Fix}_{\Lambda}(\chi_i))}_{\mathbb{Z}^6} \Bigg\backslash \mathbb{RH}^5_{\chi_i}, \quad i = 0, \dots, 4,$$

where $\mathbb{RH}^5_{\chi_i} := \mathbb{RH}\left(\mathsf{Fix}_{\Lambda}(\chi_i) \otimes_{\mathbb{Z}} \mathbb{R}\right) \cong \mathbb{RH}^5$, and $\mathsf{Stab}_{\mathbb{P}\Gamma}\left(\mathsf{Fix}_{\Lambda}(\chi_i)\right)$ can be described by the following abstract isomorphism:

$$\Gamma^{\mathbb{R}}_i := \mathsf{Stab}_{\mathbb{P}\Gamma}\left(\mathsf{Fix}_{\Lambda}(\chi_i)\right) \cong \mathbb{P}\left(\left\{ A \in \mathsf{Isom}\left(\mathsf{Fix}_{\Lambda}(\chi_i)\right) \;\middle|\; \begin{array}{c} A \text{ extends to some} \\ \text{element of } \mathsf{Isom}(\Lambda) \end{array} \right\}\right)$$

$\text{Fix}(\chi_0), \ldots, \text{Fix}(\chi_4)$

$$\begin{aligned}
\text{Fix}(\chi_0) &\cong \text{diag}(1,-1,-1,-1,-1,-1)\\
\text{Fix}(\chi_1) &\cong \text{diag}(1,-1,-1,-1,-1,-2)\\
\text{Fix}(\chi_3) &\cong \text{diag}(1,-1,-1,-2,-2,-2)
\end{aligned}$$

$$\begin{aligned}
\text{Fix}(\chi_2) &\cong \text{diag}(1,-1,-1,-1,-2,-2)\\
&\cong \text{diag}(1,-1,-1,-1,-2,-2)
\end{aligned}$$

$$\text{Fix}(\chi_4) \cong \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} -2 & 0 & 1 & 1 \\ 0 & -2 & -1 & 1 \\ 1 & -1 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}}_{\text{lattice, } \det=-4} \quad \text{or} \quad \underbrace{\begin{bmatrix} -2 & -2 & 1 & 0 & 0 & -2 \\ -2 & -6 & 3 & 0 & 0 & -4 \\ 1 & 3 & -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & -4 & 2 & -1 & 0 & -4 \end{bmatrix}}_{\det=-4}$$

NON-COMMENSURABILITY

$$\begin{array}{lll}
\text{Isom}(\text{Fix}(\chi_0)) \not\sim \text{Isom}(\text{Fix}(\chi_1)) & \text{Isom}(\text{Fix}(\chi_0)) \not\sim \text{Isom}(\text{Fix}(\chi_3)) & \text{Isom}(\text{Fix}(\chi_1)) \not\sim \text{Isom}(\text{Fix}(\chi_2))\\
\text{Isom}(\text{Fix}(\chi_1)) \not\sim \text{Isom}(\text{Fix}(\chi_4)) & \text{Isom}(\text{Fix}(\chi_2)) \not\sim \text{Isom}(\text{Fix}(\chi_3)) & \text{Isom}(\text{Fix}(\chi_3)) \not\sim \text{Isom}(\text{Fix}(\chi_4))
\end{array}$$

$\mathcal{M}_s^{\mathbb{R}} \leftrightarrow \mathbb{P}\Gamma\backslash\mathcal{K}_s$ Is Not Real Hyperbolic

Recall again:

$$\mathcal{M}_s^{\mathbb{R}} \leftrightarrow \mathbb{P}\Gamma\backslash\mathcal{K}_s = \mathbb{P}\Gamma \Bigg\backslash \left\{ \left(\coprod_{[\chi]\in\mathbb{P}\mathsf{IAAI}^{\mathbb{R}}(\Lambda)} \mathbb{RH}^5_{[\chi]} \right) \Bigg/ \approx \right\}$$

We check the local quotient structure of $\mathbb{P}\Gamma\backslash\mathcal{K}_s$ stratum by stratum.

NEGATIVE RESULT

Points in $\mathbb{P}\Gamma\backslash\mathcal{K}_s$ corresponding to the stratum $\Delta_{\mathbb{R}}^{0,1}$ (of real octics having one (real) triple point and no other singularities) can not admit a local real hyperbolic orbifold structure. Hence $\mathbb{P}\Gamma\backslash\mathcal{K}_s$ itself cannot be a real hyperbolic orbifold.

Why Points in $\Delta_{\mathbb{R}}^{0,1}$ Are Not Hyerpbolic

A point in $\Delta_{\mathbb{R}}^{0,1}$ can be locally described by $p_{0,0}(x)$, where

$$p_{a_0,a_1}(x) = (x^3 + a_1 x + a_0) \cdot r(x), \quad a_0, a_1 \in \mathbb{R}.$$

We thus examine the vanishing $(\sigma^2 = -1)$-homology of

$$y^4 = x^3 + a_1 x + a_0, \quad \text{as} \quad a_0, a_1 \to 0,$$

preserved by the action induced by $x \mapsto \overline{x}$.

Computations show that $\Lambda_0 = \begin{bmatrix} -2 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & -2 \end{bmatrix}$ has two conjugacy classes of IAAI's, say χ_1 and χ_2. We expect this since a real triple point is the limit of two kinds of smooth real 3-point configurations, namely 3 distinct real points, and one real point plus 1 complex conjugate pair. So, the local geometry of $\mathbb{P}\Gamma\backslash\mathcal{K}_s$ at a point in $\Delta_{\mathbb{R}}^{0,1}$ is given by

$$\frac{\text{Fix}(\chi_1)}{\text{Stab}_{\text{Isom}(\Lambda_0)}(\text{Fix}(\chi_1))} \bigcup \frac{\text{Fix}(\chi_2)}{\text{Stab}_{\text{Isom}(\Lambda_0)}(\text{Fix}(\chi_2))}, \quad \text{subject to certain gluing.}$$

Why Points in $\Delta_{\mathbb{R}}^{0,1}$ Are Not Hyperpbolic (Cont'd)

Local quotient at a point in $\Delta_{\mathbb{R}}^{0,1}$ is given by:

$$\frac{\mathrm{Fix}(\chi_1)}{\mathrm{Stab}_{\mathrm{Isom}(\Lambda_0)}(\mathrm{Fix}(\chi_1))} \bigcup \frac{\mathrm{Fix}(\chi_2)}{\mathrm{Stab}_{\mathrm{Isom}(\Lambda_0)}(\mathrm{Fix}(\chi_2))},$$

1. The two individual quotients above are

$$\mathbb{R}^2/(\mathbb{Z}/2 \times \mathbb{Z}/2) = \text{a } 90^\circ\text{-wedge}, \quad \text{and} \quad \mathbb{R}^2/D_4 = \text{a } 45^\circ\text{-wedge}.$$

2. The edges of the above wedges glue “pairwise.” $\Longrightarrow$ local angle is $135^\circ = 3\pi/4$.

OBSERVATION: Points in $\Delta_{\mathbb{R}}^{0,1}$ can NOT be real hyperbolic because the local anlge does not add up to $2\pi/n$, for some integer $n > 0$.

Ongoing Work & Future Directions ...

1. Prove that two representatives of IAAI's of Λ whose fixed lattice are isometric to $\mathsf{Fix}(\chi_2)$ are in fact conjugates.
2. Identify $\mathbb{P}\Gamma_4^{\mathbb{R}}$.
3. Study the topology of $\mathcal{M}_{0,i}^{\mathbb{R}}$: fundamental and higher homotopy groups.
4. Complete the examination of the local geometry of $\mathbb{P}\Gamma\backslash\mathcal{K}_s$.

   We do know that $\mathcal{K}_s$ is (obviously) a metric space and $\mathbb{P}\Gamma$ acts on it by isometries, properly discontinuously, hence with closed orbits. $\mathbb{P}\Gamma\backslash\mathcal{K}_s$ is thus itself a metric space.

Speculation:

$\mathbb{P}\Gamma\backslash\mathcal{K}_s$ *is some kind of an orbit space by a negatively curved, non-locally-symmetric space. These were once conjectured not to exist. But Mostow-Siu [1980] first constructed such compact Käher (complex) surface (hence of real dimension 4). Gromov-Thurston [1987] constructed examples of any real dimension* ≥ 4.

THE END

THANK YOU!