MATH 6720
Homework # 5
Due Date: 03/24/2017

1. Let $-\infty < a < b < \infty$ and $S(z) = \frac{z-ia}{z-ib}$. Define the lines $L_1 = \{z : \text{Im}(z) = b\}$, $L_2 = \{z : \text{Im}(z) = a\}$ and $L_3 = \{z : \text{Re}(z) = 0\}$. Determine the image under S of the six regions in the complex plane defined by the lines.

2. (a) Find the fixed points of a dilation, a translation and the inversion on $\mathbb{C}_\infty$.
   (b) Show that a Möbius transformation has 0 and ∞ as its only fixed points if and only if it is a dilation, but not the identity.
   (c) Show that a Möbius transformation has ∞ as its only fixed points if and only if it is a translation, but not the identity.

3. Solve the Laplace equation $\Delta\Phi = 0$ in the domain between the two non concentric circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 9$ with the boundary conditions $\Phi = 1$ on the inner circle and $\Phi = 2$ on the outer circle.

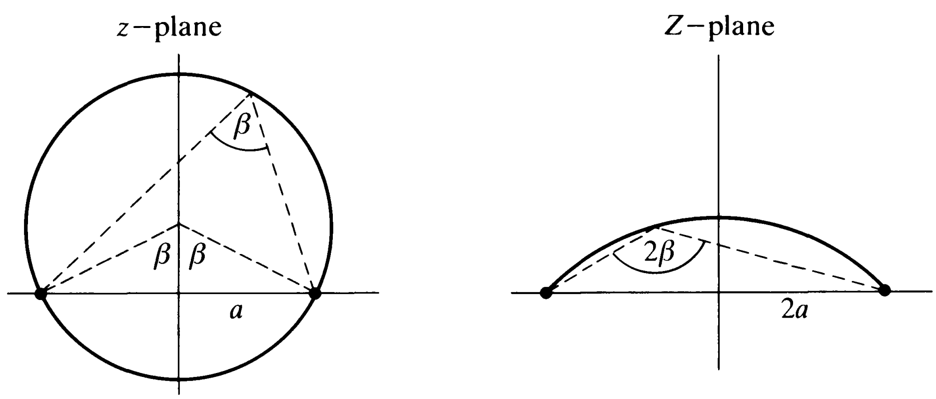
4. From Acheson.
   (a) Show that the Joukowski transformation $Z = z + a^2/z$ can be written in the form

$$\frac{Z-2a}{Z+2a} = \left(\frac{z-a}{z+a}\right)^2,$$

   so that

$$\arg(Z-2a) - \arg(Z+2a) = 2[\arg(z-a) - \arg(z+a)].$$

   (b) Consider the circle in the z-plane which passes through $z = -a$ and $z = a$ and has centre $ia\cot\beta$. Show that the above transformation takes it into a circular arc between $Z = -2a$ and $Z = 2a$ with subtended angle 2β.



   (c) Obtain an expression for the complex potential in the Z-plane, when the flow is uniform, speed U, and parallel to the real axis.
   (d) Show that the velocity will be finite at both the leading and trailing edges if

$$\Gamma = -4\pi U a \cot\beta.$$

   This exceptional circumstance arises only when the undisturbed flow is parallel to the chord line of the arc.

5. **Problem 6.5.4** Solve the generalization of Abel's integral equation

$$T(x) = \int_0^x \frac{f(y)dy}{(x-y)^\alpha}$$

for $f(x)$ in terms of $T(x)$ where $0 < \alpha < 1$.

6. **Problem 6.5.2** Verify the binomial theorem

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(n+1)} z^n.$$

For what values of α and z is this a correct formula ?

7. **Problem 6.5.9** Show that

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z.$$

8. **Problem 6.5.9** Prove that

$$J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z) = 1.$$

9. In polar coordinates, the free transverse vibrations of a stretched membrane (with equilibrium position in the $r\theta$-plane) are described by the equation

$$\Delta_p u(r,\theta,t) = \frac{1}{b^2}\frac{\partial^2 u(r,\theta,t)}{\partial t^2}$$

where Δ_p is the Laplacian in polar coordinates. Solve the equation of motion for the case of a circular membrane of radius a subject to the boundary condition $u(r=a,\theta,t)=0$ and the initial conditions $u(r,\theta,0)=f(r)$ and $\frac{\partial u(r,\theta,0)}{\partial t} = g(r)$.
HINT: Use Bessel functions.