MATH 6750
Homework # 3
Due Date: Friday, December 15th 2017 at Noon in Aaron's mailbox.

1. **Power-law flow**. Consider a film of polymer solution as it flows down an inclined plane making an angle α with the vertical. Use the power-law fluid model, $\eta = m\dot{\gamma}^{n-1}$. Take the origin of coordinates to be such that $x = 0$ at the film surface and $x = \delta$ at the plate. The film extends along the plate from $z = 0$ to $z = L$. Show that the velocity is

$$v_z = \left(\frac{\rho g \delta}{m}\cos\alpha\right)^{1/n} \frac{\delta}{1/n+1}\left[1 - \left(\frac{x}{\delta}\right)^{1/n+1}\right].$$

2. **Rouse chain at equilibrium**. Consider the Rouse chain, a freely jointed bead-spring chain with Hookean springs. The equilibrium configuration for the Rouse chain is

$$\psi_{\text{eq}} = \prod_{j=1}^{N-1}\left(\frac{H}{2\pi kT}\right)^{3/2} e^{-(H/2kT)Q_j^2}.$$

Show that the mean-square end-to-end distance is

$$< r^2 >_{\text{eq}} = \frac{3(N-1)kT}{H}.$$

3. **Hookean dummbells under steady-state shear flow**. Consider the elastic dumbbell model for Hookean dumbbells. Recall that the polymeric stress $\underline{\underline{\tau}}_p$ satisfies

$$\underline{\underline{\tau}}_p + \lambda_H \overset{\nabla}{\underline{\underline{\tau}}}_p = -nkT\lambda_H \dot{\underline{\underline{\gamma}}}$$

where $\dot{\underline{\underline{\gamma}}} = \nabla\mathbf{v} + \nabla\mathbf{v}^T$ and $\lambda_H = \frac{\zeta}{4H}$. Recall further that the total stress is $\underline{\underline{\tau}} = -\eta_s\dot{\underline{\underline{\gamma}}} + \underline{\underline{\tau}}_p$. Consider steady-state shear flow $v_x = \dot{\gamma}y, v_y = 0, v_z = 0$.

(a) Show that the solution to the polymeric stress is

$$\tau_{p,xx} = -2nkT\lambda_H^2\dot{\gamma}^2 \quad \tau_{p,xy} = \tau_{p,yx} = -nkT\lambda_H\dot{\gamma},$$

all the other $\tau_{p,ij}$ being zero.

(b) Show that the material functions are

$$\eta = \eta_s + nkT\lambda_H \quad \Psi_1 = 2nkT\lambda_H^2 \quad \Psi_2 = 0.$$

η is the viscosity defined such that $\tau_{yx} = \tau_{xy} = -\eta\dot{\gamma}$, Ψ_1 is the first normal stress coefficient defined as $\tau_{xx} - \tau_{yy} = -\Psi_1\dot{\gamma}^2$ and Ψ_2 is the second normal stress coefficient defined as $\tau_{yy} - \tau_{zz} = -\Psi_2\dot{\gamma}^2$. In other words, all material functions are constant. Therefore, the model is inadequate to describe the behavior of dilute polymer suspensions since the viscosity decreases with increasing shear rate.

4. **Numerical methods**. When we considered numerical solution of the Navier-Stokes Equations

$$\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p + \mu\Delta\mathbf{u} + \mathbf{f}, \qquad \nabla\cdot\mathbf{u} = 0, \tag{1}$$

we wrote down a semi-discrete system

$$\begin{aligned} \rho\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \mathbf{A}^{n+1/2}\right) &= -\nabla p^{n+1/2} + \frac{\mu}{2}\left(\Delta\mathbf{u}^n + \Delta\mathbf{u}^{n+1}\right) + \mathbf{f}^{n+1/2}, \\ \nabla\cdot\mathbf{u}^{n+1} &= 0. \end{aligned} \tag{2}$$

Here, $\mathbf{A}^{n+1/2}$ is an explicit in time approximation to $\mathbf{u}\cdot\nabla\mathbf{u}$ at $t = (n+1/2)\Delta t$ and $\mathbf{f}^{n+1/2}$ denotes $\mathbf{f}$ evaluated at that t. The unknowns in these equations are $\mathbf{u}^{n+1}$ and $p^{n+1/2}$. But we did not solve Eqs. **??**. Instead, we introduced the idea of a projection method such as

$$\begin{aligned} \rho\left(\frac{\mathbf{u}^{*}-\mathbf{u}^{n}}{\Delta t}+\mathbf{A}^{n+1/2}\right) &= -\nabla p^{n-1/2}+\frac{\mu}{2}\left(\Delta\mathbf{u}^{n}+\Delta\mathbf{u}^{n+1}\right)+\mathbf{f}^{n+1/2} \\ \rho\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^{*}}{\Delta t}\right) &= -\nabla p^{n+1/2} \qquad \text{with } \nabla\cdot\mathbf{u}^{n+1}=0. \end{aligned} \tag{3}$$

The intermediate field $\mathbf{u}^*$ is not divergence free but $\mathbf{u}^{n+1}$ is.

Now consider the momentum equations of a mixture model

$$\begin{aligned} \rho\left\{(\theta_{\rm f}\mathbf{u}_{\rm f})_t+\nabla\cdot(\theta_{\rm f}\mathbf{u}_{\rm f}\mathbf{u}_{\rm f})\right\} &= -\theta_{\rm f}\nabla p+\nabla\cdot(\theta_{\rm f}\underline{\underline{\sigma}}_{\rm f})+\xi\theta_{\rm f}\theta_{\rm n}(\mathbf{u}_{\rm n}-\mathbf{u}_{\rm f}), \\ \rho\left\{(\theta_{\rm n}\mathbf{u}_{\rm n})_t+\nabla\cdot(\theta_{\rm n}\mathbf{u}_{\rm n}\mathbf{u}_{\rm n})\right\} &= -\theta_{\rm n}\nabla p+\nabla\cdot(\theta_{\rm n}\underline{\underline{\sigma}}_{\rm n})+\xi\theta_{\rm f}\theta_{\rm n}(\mathbf{u}_{\rm f}-\mathbf{u}_{\rm n}), \\ \nabla\cdot(\theta_{\rm f}\mathbf{u}_{\rm f}+\theta_{\rm n}\mathbf{u}_{\rm n}) &= 0. \end{aligned} \tag{4}$$

Here, $\underline{\underline{\sigma}}_{\rm f} = 2\mu_f\left(\nabla\mathbf{u}_{\rm f}+(\nabla\mathbf{u}_{\rm f})^T\right)+\lambda(\nabla\cdot\mathbf{u}_{\rm f})\underline{\underline{I}}$ and $\underline{\underline{\sigma}}_{\rm n} = 2\mu_n\left(\nabla\mathbf{u}_{\rm n}+(\nabla\mathbf{u}_{\rm n}{}^T\right)+\lambda(\nabla\cdot\mathbf{u}_{\rm n})\underline{\underline{I}}$ are the viscous stress tensors for the fluid and network respectively. Assume that the volume fractions $\theta_{\rm f}$ and $\theta_{\rm n}$ are known and satisfy $\theta_{\rm f}+\theta_{\rm n}=1$.

(a) Write down a semi-discrete scheme for Eqs. **??** analogous to scheme **??** for the unknowns ${\mathbf{u}_{\rm f}}^{n+1}$, ${\mathbf{u}_{\rm n}}^{n+1}$, and $p^{n+1/2}$.

(b) Can you derive a projection method that allows one to avoid solving the equations in (a) simultaneously? If so, give the method. If not, explain what gets in the way.