Instructions:

- There is a two hours time limit to complete the midterm. The time limit is enforced on the honor system, do not spend the entire weekend thinking about the problems.
- Verbal or electronic collaborations are not allowed.
- Notes, books, electronic material are not allowed.

1. Conformal mappings
   (a) Show that the transformation $\xi = 1/z$ maps the line $x = c_1 \neq 0$ to a circle with center along the real axis.
   (b) A Möbius transformation maps the region between the non-concentric circles $|z| = 1$ and $|z - 13/4| = (15/4)^2$ onto an annulus $\rho_0 < |z| < 1$. Find $\rho_0$ only, i.e you don’t need to give the transformation.

2. Green’s function
   Find the Green’s function for the operator $(L - \lambda)u = \delta(x - \xi)$, $\lambda \neq 0$, $Lu = -u''$ on $[0, 1]$ with boundary conditions $u'(0) = u(1) = 0$.
   HINT: $\sin(u) \sin(v) = \frac{1}{2} [\cos(u - v) - \cos(u + v)]$ and $\cos(u) \cos(v) = \frac{1}{2} [\cos(u - v) + \cos(u + v)]$.

3. Asymptotic expansion of integrals
   Find the leading order behavior and show that the relationship is asymptotic.
   (a) $E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt \quad x \to 0^+$.
   (b) $I(x) = \int_x^\infty e^{-t^3} dt \quad x \to +\infty$.

4. Watson’s lemma
   Show that the complete asymptotic expansion of
   $$I(x) = \int_0^\infty (t^2 + 2t)^{-1/2} e^{-xt} dt$$
   is
   $$I(x) \sim \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n + 1/2)^2}{2^{n+1/2}n!\Gamma(1/2)x^{n+1/2}} \quad x \to +\infty.$$ 
   HINT: The Gamma function satisfies $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$.