Fluid Dynamics - Math 6750

Basic principles - Week 1

1 Fluids

Two of the state of all material, gas and liquid, are called fluids. Gas can be compressed much more easily than liquid, i.e. they fill the container. Fluids do not hold their shape, they flow, deform and take the shape of the surrounding container. A *free surface* is the interface between a gas and a liquid.

The most distinctive property of fluids is its response to an applied force or stress (force per area), e.g. shear or normal stresses. As long as the stress acts, a fluid element will continue to distort. It will not recover its original shape.

2 Units

We will be using the SI system of units.

Quantity	Dimension	Units
Mass	M	kilogram
Length	L	mass
Time	T	second
Velocity	L/T	m/s
Acceleration	L/T^2	m/s^2
Strain rate (Velocity gradient)	$1/T$	$1/s$
Density	M/L^3	kg/m^3
Force	ML/T^2	Newton
Energy	ML^2/T^2	Joule
Power	ML^2/T^3	Watt
Stress	$M/L/T^2$	Pascal
Viscosity	$M/L/T$	$Pa \cdot s$
Kinematic viscosity	L^2/T	m^2/s
Surface tension	M/T^2	N/m
Temperature	K	Kelvin

Table 1: Units and dimensions of common quantities.

3 The continuum approximation

There are two ways we can describe the motion of a liquid. In the first one (microscopic), we can tract the motion of individual molecules of water and use the laws of physics to describe their motions. This approach becomes computationally very expensive. For example, there are 55 moles per liter of water (H_2O) and 1 mole consists of about $6(10)^{23}$ molecules. So there are

about $330(10)^{23} = 3.3(10)^{25}$ water molecules per liter. Therefore, we adopt a second approach (macroscopic). In this case, the fluid properties will be described by continuous functions of a spatial point $\mathbf{x}$ and time t. Our goal is to derive partial differential equations (PDEs) describing the motion and properties of fluids on the continuum level.

Microscopic	*Macroscopic*
individual molecules	infinitely divisible without change of character
many-body problems	continuum
few thousands molecules	limit of the mean of a quantity over the molecular fluctuations
short time	large physical dimension
laws of dynamics	laws of mechanics and probability
$\varepsilon \sim$ mean free path	L: physical length of the fluid flow

Table 2: The continuum approximation

Any macroscopic quantity can be defined as an appropriate spatial average of molecular quantities over a small enough volume V. For example, the velocity $\mathbf{u}$ is

$$\mathbf{u} = \frac{1}{V}\int_V \mathbf{w} dV.$$

The above continuum approximation is valid if

$$\varepsilon \ll V^{1/3} \ll L.$$

On the microscopic scale, all transports phenomena stems from statistical mechanics. On the macroscopic level, transports mechanisms can be split into transport by mean of a continuum variable (e.g. $\mathbf{u}$) (also called convective processes) and transports resulting from molecular fluctuations (also called diffusive processes). We will see that molecular fluctuations lead to surface contributions and stresses.

4 Frames and flow map

Consider a region $D \in \mathbb{R}^3$ filled with a fluid.

Definition 1. Assume that at time $t = 0$, the location of a fluid particle is denoted by $\boldsymbol{\alpha}$. At time $t \geq 0$, the same fluid particle may be at location $\mathbf{x} = \Phi(\boldsymbol{\alpha}, t)$. The function $\Phi(\boldsymbol{\alpha}, t)$ is called the flow map, $\Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$.

We will assume that Φ is smooth and invertible for each t. $\Phi(\boldsymbol{\alpha}, t)$ being invertible means that given a time t and spatial location $\mathbf{x} \in D$, we can identify the unique $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} = \Phi^{-1}(\mathbf{x}, t)$. We think of $\boldsymbol{\alpha}$ as the label or color of the fluid particle. As a result, two fluid particles with $\boldsymbol{\alpha}_1 \neq \boldsymbol{\alpha}_2$ cannot be at the same $\mathbf{x}$ at the same t. Furthermore, since $\mathbf{x} = \Phi(\boldsymbol{\alpha}, t)$ is invertible, its Jacobian (determinant) is nonzero:

$$J = \left|\frac{\partial(x_1, x_2, x_3)}{\partial(\alpha_1, \alpha_2, \alpha_3)}\right| = \begin{bmatrix} \frac{\partial x_1}{\partial \alpha_1} & \frac{\partial x_1}{\alpha_2} & \frac{\partial x_1}{\partial \alpha_3} \\ \frac{\partial x_2}{\partial \alpha_1} & \frac{\partial x_2}{\alpha_2} & \frac{\partial x_2}{\partial \alpha_3} \\ \frac{\partial x_3}{\partial \alpha_1} & \frac{\partial x_3}{\alpha_2} & \frac{\partial x_3}{\partial \alpha_3} \end{bmatrix} \neq 0.$$

$|J|$ represents the dilatation of an infinitesimal volume $dV = dV(\mathbf{x})$ as it follows the motion and the change in $dV = d\mathbf{x}$ is given by the usual change of coordinates formula.

Consider a region $\Omega_0 \in D$, a region of fluid at $t = 0$. We will consider the set (image under the flow map, set of location at time t of fluid particles that were originally in Ω_0 at $t = 0$) and introduce the following notation:

$$\Omega_t = \{\mathbf{x} \in D : \mathbf{x} = \Phi(\boldsymbol{\alpha}, t) \text{ for some } \alpha \in \Omega_0\} \equiv \Omega(t) = \Phi(\Omega_0, t).$$

Ω_t is called a material volume, a volume moving with the fluid.

Definition 2 (Coordinates)**.**

1. $\boldsymbol{\alpha}$ is called a *material* or Lagrangian coordinate. It describes a particular fluid particle.

2. $\mathbf{x}$ is called a *spatial* or Eulerian coordinate. It describes a particular location in space.

For the Lagrangian description we start with a fluid element and follow it through the fluid. The Lagrangian coordinate $\boldsymbol{\alpha}$ need not be the initial position of a fluid element, although that is the most common choice. Working in the Lagrangian frame has certain theoretical and mathematical advantages, but it is often difficult to apply in practice since any measurements in a fluid tend to be performed at fixed points in space as the fluid flows past the point. In the Lagrangian system we use fluid particles, which are small parts of the fluid of fixed mass. They are called particles in analogy with the dynamics of solid bodies. We follow an individual fluid particle as it moves through the flow, and the particle is identified by its position at some initial time and the time elapsed since that instant. This particle description is the one normally used in describing the dynamics of rigid bodies because the particles tend to be few in number and easily identified. To describe a fluid flow, however, we need to follow many fluid particles, and to resolve the smallest details of the flow we may need to follow a very large ODE, such as Newton's second law, and each equation is coupled to all the others, because the motion of each particle will depend on the motion of all its neighboring particles. The solutions of these coupled ODE's are usually difficult to find because of their large number. The Lagrangian approach, therefore, is not widely used in fluid mechanics, except in some problems such as in tracking the dispersion of pollutants.

On the other hand, if we wish to observe fluid properties at a fixed location $\mathbf{x}$ as a function of time, we must realize that as time evolves different fluid elements will occupy the location x. This constitutes the Eulerian description which is the most commonly used way of describing a fluid motion. In the Eulerian system we try to find a description which gives the details of the entire flow field at any position and time. Instead of describing the fluid motion in terms of the movement of individual particles, we look for a field description. In other words, we search for a description that gives the velocity and acceleration of any fluid particle at any point at any time. At first sight, this approach appears to be very straightforward. However, we are no longer explicitly following fluid particles of fixed mass; at a given point in the flow, new particles are arriving all the time. This makes it difficult to apply Newton's second law since it applies only to particles of fixed mass. We therefore need a relationship that gives the acceleration of a fluid particle in terms of the Eulerian system. Nevertheless, the Eulerian system is generally preferred for solving problems in fluid mechanics.

Eulerian	*Lagrangian*
spatial	material, referential
fluid dynamics	elasticity
variables: $\mathbf{x}$ position in space, t time	variables: $\boldsymbol{\alpha}$ initial configuration, t time
fluid velocity: $\mathbf{u}(\mathbf{x}, t)$	fluid velocity: $\mathbf{v}(\boldsymbol{\alpha}, t)$
follows flow through a control volume fixed in space	follows identifiable pieces of matter

Table 3: Lagrangian vs Eulerian coordinate systems.

It is of interest to compare these two descriptions and explore their connections. Because the velocity at a location $\mathbf{x}$ and time t must be equal to the velocity of the fluid particle which is at this position and at this particular time, the Eulerian and Lagrangian coordinates are related as follows:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\phi(\boldsymbol{\alpha}, t), t) = \frac{\partial \phi(\boldsymbol{\alpha}, t)}{\partial t} \tag{1}$$

where $\mathbf{u}$ is the fluid velocity field. Given the Eulerian velocity field, computing Lagrangian coordinates is therefore equivalent to solving (1) with initial condition $\mathbf{x}(0) = \phi(\boldsymbol{\alpha}, 0)$.

Example 1. In one dimension, consider the velocity field given in Eulerian coordinates by $u(x, t) = \frac{2x}{1+t}$. The Lagrangian coordinate $\phi(\boldsymbol{\alpha}, t)$ can be found by solving

$$\frac{\partial \phi(\alpha, t)}{\partial t} = \frac{2\phi}{1+t} \quad \phi(\alpha, 0) = \alpha.$$

This is a separable ODE and its solution is $\phi(\alpha, t) = \alpha(1+t)^2$. The Lagrangian velocity as a function of α, t is $\frac{\partial \phi(\alpha, t)}{\partial t} = 2\alpha(1+t)$, which can also be found by evaluating the Eulerian velocity at $x = \phi(\alpha, t)$.

We introduce the following notation for derivatives with respect to time:

$$\frac{\partial}{\partial t} \equiv \left(\frac{\partial}{\partial t}\right)_{\mathbf{x} \text{ fixed}} \quad \frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{\alpha} \text{ fixed}}$$

Let $f(\mathbf{x}, t)$ be a function defined at each fluid point,where x is understood to change with time at the local fluid velocity $\mathbf{u}$, i.e. $\frac{dx_i}{dt} = u_i$, $i = 1, 2, 3$. Then $\frac{\partial f}{\partial t}$ is the rate of change of f at a fixed location $\mathbf{x}$, while $\frac{Df}{Dt}$ is the rate of change of f for the fluid particle which happens to be at location $\mathbf{x} = \Phi(\boldsymbol{\alpha}, t)$ at time t. $\frac{D}{Dt}$ is called material or Lagrangian derivative.

Remark 1. Assume that water is flowing through a pipe with a constriction and the motion is steady, i.e. the velocity at any spatial location $\mathbf{x}$ is not changing in time or $\partial_t \mathbf{u}$=0. However, since the flow rate is constant across any cross-section, $\frac{D\mathbf{u}}{Dt} \neq 0$. A sketch of this situation is given below.

Lemma 1. *Suppose $f(\mathbf{x}, t) = f(\Phi(\boldsymbol{\alpha}, t), t)$. Then, we have for the material derivative of f*

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f. \tag{2}$$

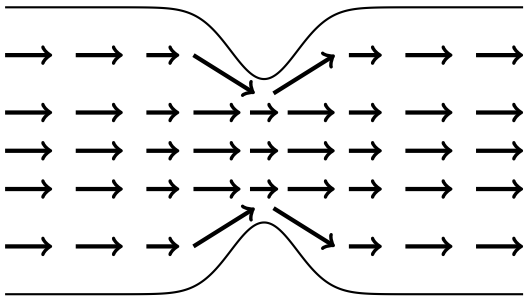

Figure 1: Steady flow in a channel with a constriction.

Proof. We simply apply chain rule to $f(\Phi(\boldsymbol{\alpha}, t), t)$:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial f} + \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t} \text{ where } \frac{\partial x_i}{\partial t} = \frac{D\Phi_i}{Dt} = u_i$$

is the component of the velocity at $(\mathbf{x}, t)$ in the ith direction. □

Lemma 2 (Euler)**.**

$$\frac{DJ}{Dt} = J\nabla \cdot \mathbf{u} \tag{3}$$

5 Kinematics

Kinematics refers to a description of motion. It is not concerned with what causes the motion. For a fluid, we will have a velocity field $\mathbf{u}(\mathbf{x}, t)$ defined for each spatial location $\mathbf{x}$ in the fluid and each time t. Several sets of curves related to the velocity field play a role in fluid dynamics.

Definition 3 (Flow lines)**.**

1. The flow is *steady* if $\mathbf{u}$ is independent of t, i.e $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x})$
2. A *streamline* is a curve which is everywhere tangent to the velocity field, $\mathbf{u}(\mathbf{x}, t)$.
3. A *particle path* consists of the set of points occupied by a given fluid particle as it moves in time.
4. A *streakline* is a curve which, at time t, consists of the locations of fluid particles that passed through a given location $\mathbf{x}_0$ at some earlier time.
5. A flow flield is 2D when $\mathbf{u}(\mathbf{x}, t)$ is everywhere perpendicular to a certain direction and independent of displacements parallel to that direction.

6. A flow field is axisymmetric when $\mathbf{u}$ expressed in cylindrical coordinates is independent of the azimuthal angle θ.

Remark 2.

- If a curve is described by $(x(s), y(s), z(s))$ for some parameter s, then that curve is a streamline if

  $$\frac{dx}{ds} = u(x(s), y(s), z(s), t) \quad \frac{dy}{ds} = v(x(s), y(s), z(s), t) \quad \frac{dz}{ds} = w(x(s), y(s), z(s), t) \quad \text{for all } t.$$

  Note that t enters here as a parameter. Streamlines are defined at each time t relative to the velocity field that exists at that time.

- Suppose the particle is at location (x_0, y_0, z_0) at $t = 0$. The corresponding particle path is the curve which satisfies

  $$\frac{dx}{dt} = u(x(t), y(t), z(t), t), \quad \frac{dy}{dt} = v(x(t), y(t), z(t), t), \quad \frac{dz}{dt} = w(x(t), y(t), z(t), t)$$

  with the conditions $x(0) = x_0, y(0) = y_0, z(0) = z_0$. The particle path is the trajectory of a given fluid particle.

- The path of a particle coincide with the streamline only if the flow is steady.

- Physically, one can make a streak line by continuously injected dyed fluid at a fixed location in the fluid. At time t, the curve traced by dyed particles is the streakline at time t through the injection point $\mathbf{x}_0$. The streakline consists of the set of points $\mathbf{x}(t)$ that satisfies

  $$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}(t), t), \mathbf{x}(\tau) = \mathbf{x}_0 \text{ for some } \tau \leq t.$$

Definition 4 (Circulation and Vorticity)**.**

1. The circulation Γ is defined as $\Gamma = \oint \mathbf{u} \cdot d\boldsymbol{\gamma}$, where the contour integral is taken counter-clockwise.

2. The vorticity $\boldsymbol{\omega}$ is the curl of $\mathbf{u}$: $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

6 Tensor Calculus (Cartesian)

The physical quantities encountered in fluid mechanics can be divided into three categories:

1. **scalars** (zero-order tensors) such as shear rate, energy, volume and time;

2. **vectors** (first-order tensors) such as velocity, momentum and force;

3. **second-order tensors** such as stress and rate of strain tensors.

The cartesian basis vectors are $\mathbf{e}_i$, $i = 1, 2, 3$, for example $\mathbf{e}_1 = [1\,0\,0]^T$. A vector is given by $\mathbf{u} = u_i \mathbf{e}_i$. Summation over the range of an index is understood when repeated dummy indices appear in the same multiplicative term, also called Euler summation.

We start with a few basic facts about the Kronecker Delta and the permutation symbol.

Definition 5 (Kronecker Delta)**.**

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} .$$

A *permutation* is the act of rearranging members of a given set into some other order. A *transposition* is a permutation in which two adjacent indices are interchanged. An *even* (*odd*) permutation is a permutation that can be achieved in an even (odd) number of transpositions.

Definition 6 (Permutation symbol, Levi-Civita symbol)**.**

$$\varepsilon_{mnr} = \begin{cases} 0 & \text{when any two indices are equals} \\ 1 & \text{when } m,n,r \text{ are } 1,2,3 \text{ or an even permutation of } 1,2,3 \\ -1 & \text{when } m,n,r \text{ are an odd permutation of } 1,2,3 \end{cases} .$$

Proposition 1.

1. $\delta_{mm} = 3 \quad \delta_{mn}\delta_{mn} = 3.$
2. *(Epsilon-Delta Identity)*

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \quad \varepsilon_{mjk}\varepsilon_{njk} = 2\delta_{mn}.$$

Remark 3. The determinant of a 3×3 matrix A is also related to the permutation symbol:

$$\det(A) = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k} = |\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)|,$$

where $\mathbf{a}_i$ is the ith column of the matrix A.

Vector operations like dot and cross products are defined on the basis vector.

Definition 7 (Operations)**.** The dot and cross products of basis vectors are

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad \text{and} \quad \mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\mathbf{e}_k.$$

Remark 4. From the above definitions, it is easy to see that for two vectors $\mathbf{u}, \mathbf{v}$

$$\mathbf{u} \cdot \mathbf{v} = u_i\mathbf{e}_i \cdot v_j\mathbf{e}_j = u_iv_j\delta_{ij} = u_iv_i \quad \text{and} \quad \mathbf{u} \times \mathbf{v} = \varepsilon_{ijk}u_iv_j\mathbf{e}_k.$$

Example 2. Prove the following identity involving the triple vector product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

We have

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (a_m\mathbf{e}_m) \times (b_ic_j\varepsilon_{ijk}\mathbf{e}_k) = a_mb_ic_j\varepsilon_{ijk}\varepsilon_{mkn}\mathbf{e}_n \\ &= -a_mb_ic_j\varepsilon_{ijk}\varepsilon_{mnk}\mathbf{e}_n = -a_mb_ic_j\left[\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right]\mathbf{e}_n \\ &= -a_ib_ic_n\mathbf{e}_n + a_jc_jb_n\mathbf{e}_n = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \end{aligned}$$

We will use the next fact about δ_{ij} and derivatives in many derivations.

Proposition 2. *Let* $\mathbf{x}$ *be a position vector with coordinates* x_i*. Then*

$$\delta_{ij} = \frac{\partial x_i}{\partial x_j}.$$

6.1 Tensor algebra

Vectors are examples of rank one tensors. Matrices are example of second-rank tensors, generally just called tensors. A rank zero tensor is simply a scalar.

Definition 8 (Rank two tensor)**.** A second-rank tensor $\mathbf{T}$ represents a linear vector function associating with each vector $\mathbf{u}$ another vector $\mathbf{v}$, i.e $\mathbf{u} = \mathbf{T} \cdot \mathbf{v}$. The dyadic form of a tensor $\mathbf{T}$ is $\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$.

There are three basics operations on tensors. The inner and double inner product $(\cdot, :)$ are contractions and reduce the rank of tensor. The tensor or outer product $(\otimes)$ increases the rank of the tensor. More specifically, let $\mathbf{u}, \mathbf{v}$ be vectors and $\mathbf{T}, \mathbf{U}$ be tensors.

1. The inner product $\cdot$ which retains or reduces the rank of the tensors;

$$\begin{aligned}
\mathbf{T} \cdot \mathbf{u} &= (T_{ij}\mathbf{e}_i\mathbf{e}_j) \cdot (u_k\mathbf{e}_k) &&= T_{ij}u_k\delta_{jk}\mathbf{e}_i &&= T_{ij}u_j\mathbf{e}_i && \text{(Rank 1)}\\
\mathbf{u} \cdot \mathbf{T} &= (u_i\mathbf{e}_i) \cdot (T_{jk}\mathbf{e}_j\mathbf{e}_k) &&= T_{jk}u_i\delta_{ij}\mathbf{e}_k &&= T_{jk}u_j\mathbf{e}_k && \text{(Rank 1)}\\
\mathbf{T} \cdot \mathbf{S} &= (T_{ij}\mathbf{e}_i\mathbf{e}_j) \cdot (S_{kl}\mathbf{e}_k\mathbf{e}_l) &&= T_{ij}S_{kl}\delta_{jk}\mathbf{e}_i\mathbf{e}_k &&= T_{ij}S_{jl}\mathbf{e}_i\mathbf{e}_l. && \text{(Rank 2)}
\end{aligned}$$

   These correspond to standard matrix-vector and matrix-matrix multiplication.

2. The double inner product or *contraction* : which reduces the rank of the tensors;

$$\begin{aligned}
\mathbf{T} : \mathbf{S} &= T_{ij}S_{ij} && \text{(Rank 0)}\\
\varepsilon : T &= \varepsilon_{ijk}T_{jk}\mathbf{e}_i = \varepsilon_{jki}T_{jk}\mathbf{e}_i = \varepsilon_{kij}T_{jk}\mathbf{e}_i. && \text{(Rank 1)}
\end{aligned}$$

3. The outer product or *tensor product* $\otimes$ which retains or increases the rank of the tensor;

$$\mathbf{u} \otimes \mathbf{T} = (u_i\mathbf{e}_i) \otimes (T_{jk}\mathbf{e}_j\mathbf{e}_k) = u_iT_{jk}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k. \qquad \text{(Rank 3)}$$

Remark 5. $\mathbf{e}_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j$ form the unit basis for the second rank tensors with respect to the scalar product :. It is a nine dimensional vector space.

Remark 6. The cross product is not a general tensor operation as it is only defined between two rank one tensors.

Definition 9. A tensor is symmetric if $T_{ij} = T_{ji}$. A tensor is antisymmetric (also called skew symmetric) if $T_{ij} = -T_{ji}$.

Definition 10. A tensor is isotropic if its components are invariant with respect to all possible rotations of the coordinate system. That is if the quantity represented by the tensor is a function of position only.

Remark 7. All tensors of rank 0 are isotropic, while all tensors of rank 1 are not (vectors have a direction).

Theorem 1 (Isotropic tensors)**.** *The only isotropic tensors of order 2, 3, 4 are of the form*

$$\begin{aligned}
&\textit{rank } 2: && \alpha\delta_{ij}\\
&\textit{rank } 3: && \alpha\varepsilon_{ijk}\\
&\textit{rank } 4: && \alpha\delta_{ij}\delta_{pq} + \beta(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) + \gamma(\delta_{ip}\delta_{jp} - \delta_{iq}\delta_{jp}),
\end{aligned}$$

where α, β, γ are scalars.

Proof. We only give the proof for rank 2 tensors. Consider a general rank 2 tensor $\mathbf{T}$ with components T_{ij} with respect to some coordinate frame $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ and suppose that it is isotropic. Consider the 90° counter-clockwise rotation about the $\mathbf{f}_3$-axis and $\mathbf{f}_2$-axis, which can be expressed in terms of the second-order tensor $\mathbf{S}_3$ and $\mathbf{S}_2$ respectively:

$$\mathbf{S}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Since $\mathbf{T}$ is isotropic, the following must be true:

$$\mathbf{S}_3\,\mathbf{T}\mathbf{S}_3^T = \mathbf{T} = \mathbf{S}_2\,\mathbf{T}\,\mathbf{S}_2^T.$$

Expanding these yields

$$\begin{bmatrix} T_{22} & -T_{21} & -T_{23} \\ -T_{12} & T_{11} & T_{13} \\ -T_{32} & T_{31} & T_{33} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} T_{33} & T_{32} & -T_{31} \\ T_{23} & T_{22} & -T_{21} \\ -T_{13} & -T_{12} & T_{11} \end{bmatrix}.$$

Comparing the first two matrices, we see that $T_{11} = T_{22}$ and

$$\begin{aligned} -T_{23} = T_{13} = T_{23} &\implies T_{23} = T_{13} = 0 \\ -T_{32} = T_{31} = T_{32} &\implies T_{32} = T_{31} = 0. \end{aligned}$$

Comparing the last two matrices, we see that $T_{11} = T_{33}$ and $T_{12} = T_{32} = 0$, $T_{21} = T_{23} = 0$. Therefore all the off-diagonal elements of $\mathbf{T}$ are zero and all the diagonal elements are equal, say α. The claim follows. □

6.2 Tensor calculus

All of the previous operations are algebra operations. Since tensors represent quantities that are changing in space and time, we can define tensor calculus operations like we did in multivariable Calculus. The main operator is the nabla or del operator, which we define next. For the moment, we will define everything with respect to the Cartesian coordinates system and leave the generalization to spherical and cylindrical coordinates for later.

Definition 11 (Del operator)**.**

$$\nabla = \mathbf{e}_i \frac{\partial}{\partial i} = \mathbf{e}_i \partial_i.$$

We can now apply tensor operations between ∇ and a general tensor field. The most important ones are the gradient and the divergence.

Definition 12 (Gradient)**.** The gradient of tensor field $\mathbf{A}$ of any rank is

$$\nabla \mathbf{A} = \nabla \otimes \mathbf{A}.$$

Example 3. Let ϕ be a scalar field and $\mathbf{u}$ a vector field. Then

$$\begin{aligned} \nabla\phi = \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\partial \phi}{\partial x_k}\mathbf{e}_k = \partial_k \phi \mathbf{e}_k, \\ \nabla \mathbf{u} = \mathbf{e}_i \partial_i u_j \mathbf{e}_j = \partial_i u_j \mathbf{e}_i \mathbf{e}_j. \end{aligned}$$

Definition 13 (Divergence)**.** The divergence of a tensor field $\mathbf{A}$ of any rank is

$$\mathrm{div} = \nabla \cdot \mathbf{A}.$$

Example 4. Let $\mathbf{u}$ be a vector field and $\mathbf{T}$ be a tensor field. Then

$$\begin{aligned} \mathrm{div}(\mathbf{u}) = \nabla \cdot \mathbf{u} = \mathbf{e}_i \partial_i \cdot u_j \mathbf{e}_j = \partial_i u_j \delta_{ij} = \partial_i u_i, \\ \mathrm{div}(\mathbf{T}) = \nabla \cdot \mathbf{T} = \mathbf{e}_i \partial_i \cdot T_{jk} \mathbf{e}_j \mathbf{e}_k = \partial_i T_{jk} \delta_{ij} \mathbf{e}_k = \partial_i T_{ik} \mathbf{e}_k. \end{aligned}$$

Remark 8. $\nabla \mathbf{u}$ is sometimes denoted $\overrightarrow{\nabla} \mathbf{u}$ to avoid confusion with $\overleftarrow{\nabla}$ defined as: $\mathbf{u} \overleftarrow{\nabla} = u_i \mathbf{e}_i \partial_j \mathbf{e}_j = \partial_j u_i \mathbf{e}_i \mathbf{e}_j$.

Since ∇ is a rank one tensor, we can take its cross product with a vector field.

Definition 14 (Curl)**.** The curl (or rotation) of a vector field $\mathbf{u}$ is

$$\mathrm{curl}(\mathbf{u}) = \nabla \times \mathbf{u} = \mathbf{e}_i \partial_i \times u_j \mathbf{e}_j = \partial_i u_j \varepsilon_{ijk} \mathbf{e}_k.$$

Definition 15. A vector field is irrotational if $\mathrm{curl}(\mathbf{u}) = 0$.

Taking the inner product of ∇ with itself is called the Laplace operator Δ.

Definition 16 (Laplace operator)**.**

$$\Delta = \nabla^2 = \nabla \cdot \nabla.$$

Example 5. The Laplacian of a scalar field $\phi(\mathbf{x})$ is $\nabla \cdot (\nabla \phi) = \Delta \phi$.

Example 6. For a vector field $\mathbf{u}(\mathbf{x})$, the vector Laplacian is

$$\Delta \mathbf{u} = \nabla \cdot \nabla \mathbf{u} = (\partial_i \mathbf{e}_i) \cdot (\partial_j u_k \mathbf{e}_j \mathbf{e}_k) = \partial_i \partial_j u_k \delta_{ij} \mathbf{e}_j = \partial_i \partial_i u_k \mathbf{e}_k,$$

i.e $\Delta \mathbf{u} = (\Delta u_1, \Delta u_2, \Delta u_3)^T$.

The next proposition summarizes different relations between algebra and calculus operations.

Proposition 3. *Let ϕ be a scalar field and $\mathbf{u}$ be a vector field. Then*

1. $\nabla \cdot (\phi \mathbf{u}) = \nabla \phi \cdot \mathbf{u} + \phi \nabla \cdot \mathbf{u}$.
2. $\nabla \times (\nabla \phi) = 0$.
3. $\nabla \cdot (\nabla \times \mathbf{u}) = 0$.
4. $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$.

Theorem 2 (Divergence Theorem)**.** *Let V be a Lipschitz domain in $\mathbb{R}^3$ with piecewise smooth boundary $\partial V = S$. If $\mathbf{u}$ is a C^1 vector field defined on a neighborhood of V, then*

$$\int_V \nabla \cdot \mathbf{u}\, dV = \int_S \mathbf{u} \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the outward unit normal vector on S. The integral on the right is the flux *of $\mathbf{u}$ across the boundary (oriented surface) S.*

We can generalized the divergence theorem to second-order tensor fields.

Theorem 3 (Generalized Divergence Theorem)**.** *Let V be a Lipschitz domain in $\mathbb{R}^3$ with piecewise smooth boundary $\partial V = S$. If $\mathbf{T}$ is a second-order tensor field defined on a neighborhood of V, then*

$$\int_V \nabla \cdot \mathbf{T}\, dV = \int_S \mathbf{n} \cdot \mathbf{T}\, dS,$$

where $\mathbf{n}$ is the outward unit normal vector on S.

Proof. Let $\mathbf{n} = (n_1, n_2, n_3)^T$. The main idea is to apply the divergence theorem to each component of $\nabla \cdot \mathbf{T}$:

$$\int_V \nabla \cdot \mathbf{T}\, dV = \int_V \partial_j T_{jk} \mathbf{e}_k\, dV = \int_A n_j T_{jk} \mathbf{e}_k\, dA = \int_A \mathbf{n} \cdot \mathbf{T}\, dA.$$

For the last equality, note that

$$\mathbf{n} \cdot \mathbf{T} = (n_j \mathbf{e}_j) \cdot (T_{ik} \mathbf{e}_i \mathbf{e}_k) = n_j T_{ik} \delta_{ji} \mathbf{e}_k = n_j T_{jk} \mathbf{e}_k.$$

□