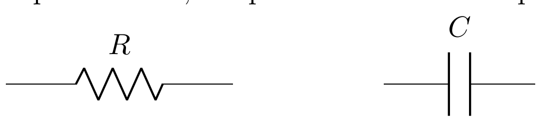
Fluid Dynamics - Math 6750

Linear Viscoelasticity

We are looking for a simple model of a fluid that behaves like a fluid in some "limit" and a solid in another "limit". The constitutive relationship will depend from the past history but will be linear. We start by an analogy to electrical circuit.

1 Electrical circuits

Definition 1. A resistor, characterized by its resistance R, is a passive electrical component that implements electrical resistance as a circuit element. A capacitor, characterized by its capacitance C, is a passive electrical component that stores electrical energy in an electric field.



Remark 1. A resistor dissipate power in the form of heat, but power flows in and out of a capacitor. The hydraulic analogy to a capacitor is a rubber membrane sealed inside a pipe. Water molecules can't pass through the membrane, but some water can move by stretching the membrane. The hydraulic analogy of a resistor is that of pushing water through a clogged pipe. It requires a harder push (pressure, voltage) to achieve the same flow (current).

Remark 2. A resistor obeys Ohm's law: $I = \frac{1}{R}V$, where V is the voltage and I the current. A capacitor obeys $I = C\frac{dV}{dt}$.

A RC circuit is an electrical circuit consisting of a capacitor and a resistor either in series (left) or in parallel (right). The schematic of the circuits is given in Fig. 1.

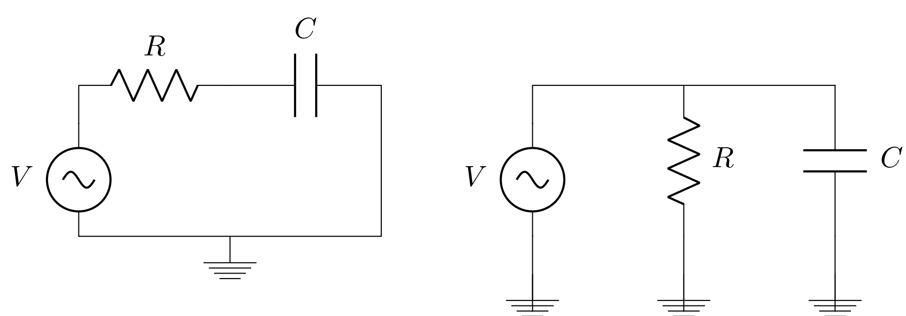


Figure 1: Series RC circuit (left) and Parallel RC circuit (right).

In a series RC circuit, the current is the same through the circuit, $I = I_C = I_R$, but the voltage is not. Moreover, from Kirchhoff's voltage law, the sum of the voltage drops must equal the applied voltage:

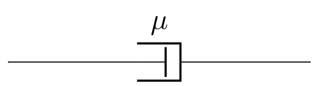
$$V = V_C + V_R.$$

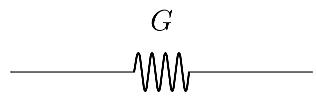
The resistor voltage is in phase with the current, while the capacitor voltage lags the current by 90°. In a parallel RC circuit, the voltage is the same, $V = V_C = V_R$, while the total current is split into the current through the resistor and the current through the capacitor:

$$I = I_C + I_R.$$

2 Mechanical model

Definition 2. A dashpot (damper), characterized by its impedance μ, is a mechanical device that resists motion via friction. A spring, characterized by its modulus G, is a mechanical device which acts to resist displacement.





Remark 3. The dashpot is the equivalent of a resistor, while a spring is the equivalent of a capacitor. A spring obeys Hooke's law, i.e. the force is proportional to the displacement. For a dashpot the force is proportional to the velocity.

Written in terms of stress instead of forces, the relationships are

$$\sigma = \mu\dot{\gamma} \text{ dashpot} \quad \sigma = G\gamma \text{ spring}$$

In this sense, a dashpot models a Newtonian fluid, while a spring models a Hookean elastic material. Here $\dot{\gamma}$ is the velocity gradient (strain rate), while γ is the displacement gradient (strain). In the analogy with electrical circuits the force (and hence the stress) corresponds to the current, $\sigma \leftrightarrow I$, while the strain corresponds to the voltage, $\gamma \leftrightarrow V$.

We now consider the combination of a spring and a dashpot in series (left) and in parallel (right). The schematic of both configurations is given in Fig. 2. From the analogy with

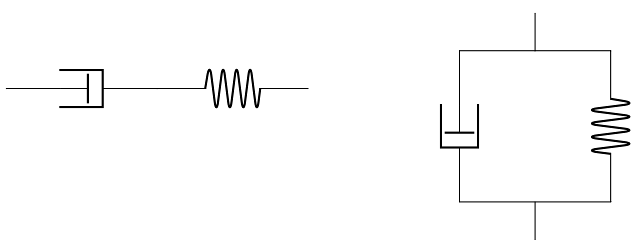


Figure 2: Spring and dasphot in series (left) and in parallel (right).

electrical circuits, we know that in a parallel spring dashpot circuit $\gamma = \gamma_s = \gamma_d$ and $\sigma = \sigma_s + \sigma_d$, while in a series spring dashpot circuit $\sigma = \sigma_s = \sigma_d$ and $\gamma = \gamma_s + \gamma_d$.

Definition 3. A linear Maxwell fluid is the analogous to a series spring-dashpot model. A linear Kelvin-Voigt viscoelastic solid is the analogous to a parallel spring-dashpot model.

Remark 4. To explain the distinction between fluid and solid, consider both types of viscoelastic elements subjected to a fixed displacements of their ends. In the serial connection, both the spring and the dashpot are stretched initially. However, the displacement of the spring can be redistributed to the dashpot, keeping the total displacement constant, and resulting in the absence of stress in this material at long times, since $\sigma \sim \dot{\gamma}$ for a dashpot. On the contrary, the parallel connection remains under stress for as long as $\gamma \neq 0$.

For the Kelvin-Voigt model in parallel, we have for the total stress

$$\sigma = \sigma_s + \sigma_d = G\gamma_s + \mu\dot{\gamma}_d = G\gamma + \mu\dot{\gamma}.$$

For the Maxwell model in series, we have after taking the time derivative of γ:

$$\dot{\gamma} = \dot{\gamma}_s + \dot{\gamma}_d = \frac{\sigma_s}{\mu} + \frac{\dot{\sigma}_d}{G} = \frac{\sigma}{\mu} + \frac{\dot{\sigma}}{G}.$$

Rearranging the last equation and letting $\lambda = \mu/G$, we have the non-homogeneous ODE for the total stress

$$\sigma + \lambda\dot{\sigma} = \mu\dot{\gamma}.$$

The solution obtained via the variation of constants is

$$\sigma(t) = G\int_{-\infty}^{t} e^{-\frac{(t-s)}{\lambda}}\dot{\gamma}(s)ds. \tag{1}$$

Integrating by parts, the above integral can be rewritten as

$$\sigma(t) = \frac{G}{\lambda}\int_{-\infty}^{t} e^{-\frac{(t-s)}{\lambda}}\gamma(t,s)ds,$$

where $\gamma(t,s) = \int_s^t \dot{\gamma}(t')dt'$ is the strain accumulated between s and t.

To understand that the Maxwell model gives rise to both an elastic and a viscous response, we consider the periodic strain deformation $\gamma(t) = \gamma_0 \sin(\omega t)$. Plugging into Eq. (1) and using a table of integrals, we find

$$\sigma(t) = \gamma_0\mu\omega\frac{\cos(\omega t) + \lambda\omega\sin(\omega t)}{1+(\lambda\omega)^2} = \frac{\mu}{1+(\lambda\omega)^2}\dot{\gamma}(t) + G\frac{(\lambda\omega)^2}{1+(\lambda\omega)^2}\gamma(t).$$

In other words, the stress can be decomposed into a component that is in phase with the strain (elastic) and a component that is in phase with the strain rate (viscous, out-of-phase with the strain). We denote by $\tilde{\mu}(\omega) = \frac{\mu}{1+(\lambda\omega)^2}$ the frequency depending viscosity and by $\tilde{G}(\omega) = G\frac{(\lambda\omega)^2}{1+(\lambda\omega)^2}$ the frequency dependent shear modulus. At large frequencies (short times), $\lambda\omega \gg 1$, $\tilde{G}(\omega) \approx G$ and $\tilde{\mu}(\omega) \approx 0$ and the response is solid like. At short frequencies (long times), $\lambda\omega \ll 1$, $\tilde{G}(\omega) \approx 0$ and $\tilde{\mu}(\omega) \approx \mu$ and the response is fluid like.

3 Frame invariance

The principle of frame invariance states that a constitutive relationship must be the same in every frame, in particular it must be independent of rigid body rotation. In order to generalize Eq. (1) to a tensor equation, we first must find the frame invariant quantity to replace γ with.