A Variational Characterization of Fluid Sloshing with Surface Tension

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Introduction

Sloshing refers to the free surface motion of a fluid in a container. We study the linearized sloshing problem of an incompressible, inviscid, irrotational fluid in a container, where we focus on the regime where the effect of surface tension dominates the effect of gravity. This is especially important in the microgravity environment, where scientists and engineers have worked to improve our understanding of the behavior of a liquid propellant within a rocket. We are interested in studying the properties of sloshing frequencies and the corresponding modes.

Surface Tension

Assume that the free surface is a graph $z = \eta(x, y, t)$. Surface tension introduces a pressure jump across the free surface:

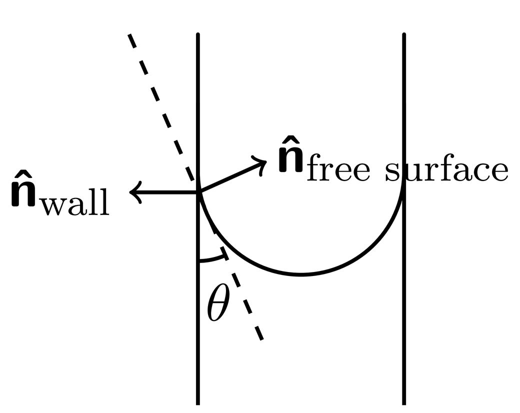
$$p - p_{\text{atm}} = -2\,T\,H$$

$T =$ Surface tension
$H =$ Mean curvature

H has terms involving η_{xx}, η_{yy}, thus it requires additional boundary conditions, commonly referred to as the contact-line boundary condition (CLBC).

CLBC and Contact Angle

The contact-line boundary condition captures the contact line behavior and the change in contact angle θ.



$$\hat{\mathbf{n}}_{\text{wall}} \cdot \hat{\mathbf{n}}_{\text{free surface}} = \cos\theta.$$

↕

Geometric equation for the contact angle

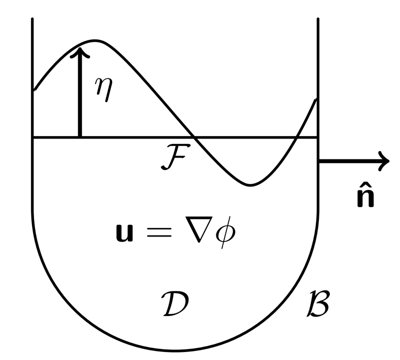
- *Free-end* **(Neumann)**: $\partial_{\hat{\mathbf{n}}}\eta = 0$
- *Pinned-end* **(Dirichlet)**: $\eta_t = 0$
- *Intermediate* **(Robin)**: $\eta_t = \lambda \partial_{\hat{\mathbf{n}}}\eta$

Assumptions:
1. Contact line freely slips.
2. Contact angle remains unchanged.
3. Contact angle equals $90°$.

Linearized Sloshing Problem

Look for time harmonic solutions.

$$\begin{aligned}
\Delta\phi &= 0 && \text{in } \mathcal{D}\\
\partial_{\hat{\mathbf{n}}}\phi &= 0 && \text{on } \mathcal{B}\\
\phi_z &= \omega\,\eta && \text{on } \mathcal{F}\\
\eta - \frac{1}{\text{Bo}}(\eta_{xx} + \eta_{yy}) &= \omega\,\phi && \text{on } \mathcal{F}\\
\partial_{\hat{\mathbf{n}}}\eta &= 0 && \text{on } \partial\mathcal{F}.
\end{aligned}$$



Eigenvalues on BCs
Generalized Steklov problem!

$$\text{Bo} = \text{Bond number} = \frac{\rho g L^2}{T} = \frac{\text{gravitational force}}{\text{surface tension force}}$$

Variational Formulation

Dirichlet energy: $D(\phi) = \dfrac{1}{2}\displaystyle\int_{\mathcal{D}} |\nabla\phi|^2\, dV$

Free surface energy: $S(\eta) = \dfrac{1}{2}\displaystyle\int_{\mathcal{F}} \left(\eta^2 + \frac{1}{\text{Bo}}|\nabla_{\mathcal{F}}\eta|^2\right) dA$

Define the function space:

$$\mathbf{H} = \{(\phi, \eta) \in H^1(\mathcal{D}) \times H^1(\mathcal{F}) \colon [\phi]_{\mathcal{F}} = 0 = [\eta]_{\mathcal{F}}\}$$

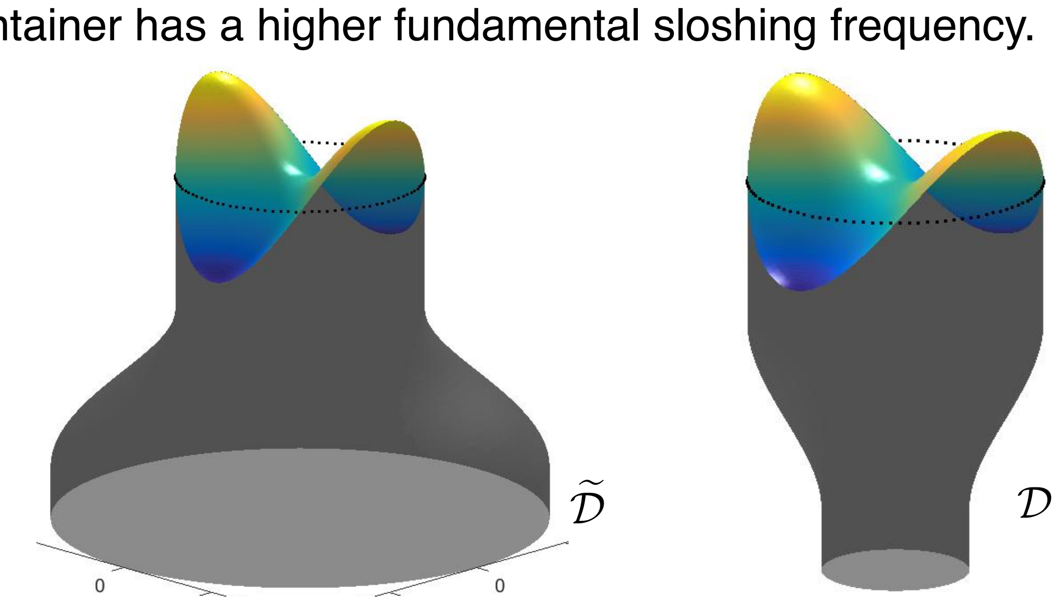
The fundamental sloshing frequency has the following variational characterization:

$$\inf_{(\phi,\eta)\in\mathbf{H}} D(\phi) + S(\eta) \quad \text{subject to} \quad \int_{\mathcal{F}} \phi\eta\, dA = 1.$$

- Rayleigh-Ritz generalization for higher sloshing frequency.
- In the limit as $\text{Bo} \longrightarrow \infty$, we recover the variational principle for the linearized sloshing problem neglecting surface tension.

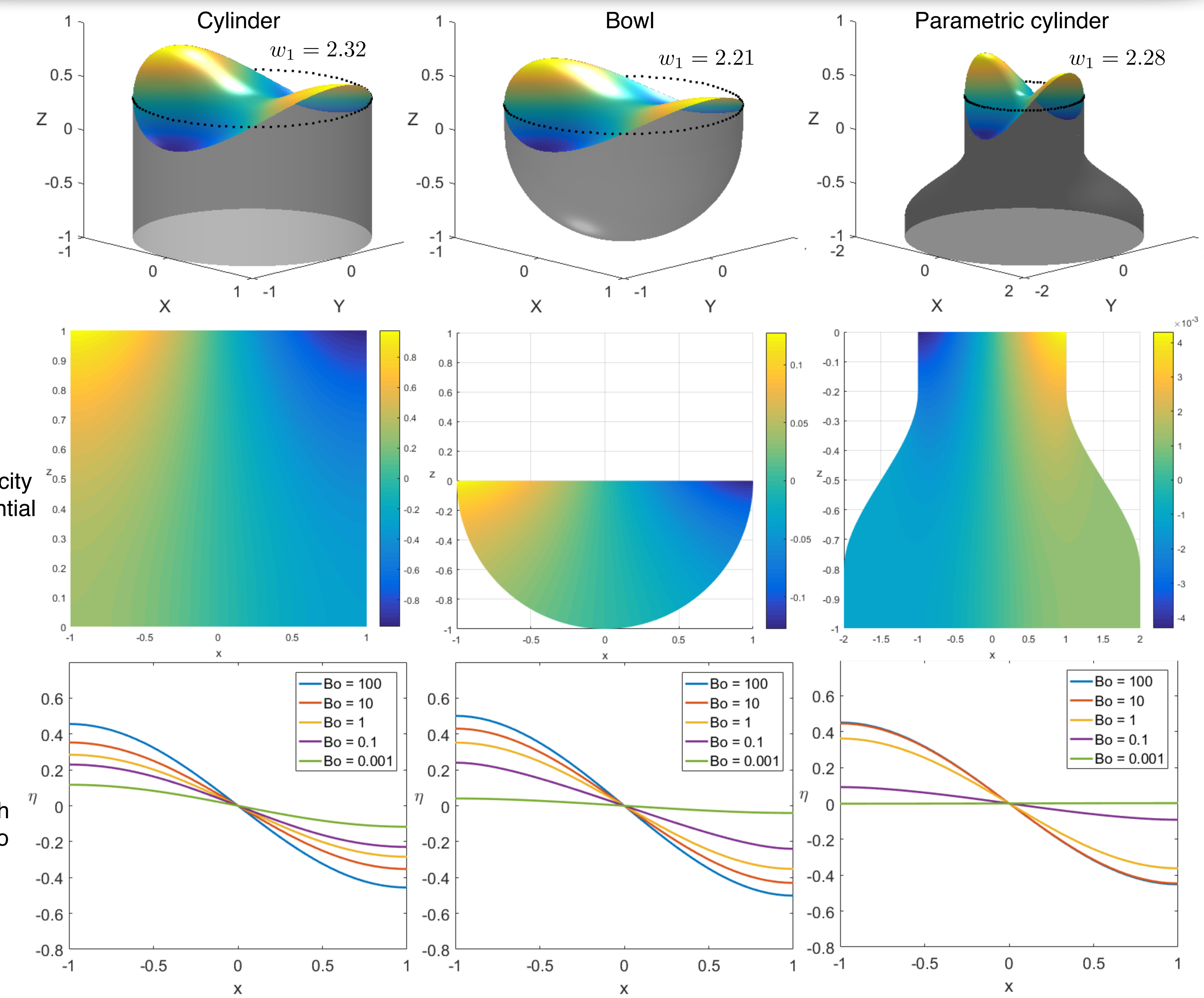
Domain Monotonicity

For two containers having identical free surface and container walls which are both vertical at the free surface, the larger container has a higher fundamental sloshing frequency.



$$w_1(\tilde{\mathcal{D}}) = 2.28 > 2.19 = w_1(\mathcal{D})$$

Numerical Results



Discussion

Relatively simple coupled systems, but requires the following assumptions:
- $90°$ contact angle.
- Vertical wall near the free surface.
- Contact angle does not change.

1. Are there better ways to handle a curved free surface? Perhaps use curvilinear coordinates?
2. If we have a curved wall at the boundary of a free surface, how to linearize the system correctly?
3. Consider the case where the contact angle changes over time, is it possible to determine the correct contact-line boundary condition?

Future Directions

1. **Isoperimetric problem**: for a fixed volume, determine the shape of axisymmetric container that maximizes the fundamental sloshing frequency.
2. **Isochronous problem**: find containers such that the fundamental sloshing frequency is independent of the level to which the container is filled.
3. **High spots**: determine the location of the maximum elevation of the free surface.

References

1. C. H. Tan, C. Hohenegger and B. Osting, A variational characterization of fluid sloshing with surface tension, *to appear in the SIAM Journal of Applied Mathematics (2017).*
2. C. H. Tan, M. Carlson, C. Hohenegger and B. Osting, A finite element approach for fluid sloshing with surface tension, *in preparation (2017).*