Final Exam
M 5500, Spring 2018

Solve any six problems

Return at Monday, April 30, before 5 pm

1. Find Euler equation and natural boundary condition, check Legendre condition for the isoperimetric problem

$$I(u) = \min_{u(x)} \int_0^1 \left((u')^2 - 3u\,u' + 3\sin(x)u^2\right) dx \quad \text{if} \int_0^1 (u)\,dx = 1$$

2. Two equal gravitating masses $m = 1$ with coordinates $r_1(t)$ and $r_2(t)$ move in a plane,

$$r_i(t) = (x_i(t), y_i(t))$$

Gravitational force is:

$$F = \gamma \frac{r_1 - r_2}{|r_1 - r_2|^3}$$

Derive kinetic and potential energy, formulate a variational problem. Find Lagrangian, Hamiltonian, and equations of motion of the masses. Find invariants to the movement.

Hint: Express coordinates through the center of masses $r_0 = \frac{1}{2}(r_1+r_2)$ and deviation from the center.

3. Consider the problem

$$I = \min_{u(x)} \int_0^1 [a\,(u')^2 - c\,u(u-1)]dx, \quad u(0) = 0$$

a. Write Hamiltonian and dual Lagrangian.
b. Derive Euler equations and boundary conditions for the primary and the dual problems, derive system of first-order equation and boundary conditions.

4. A process is described by a differential equation

$$-au'' + cu^3 + 1 = 0, \quad \text{in } (0,1) \quad u(0) = 0, \; u(1) + u'(1) = 3$$

Write a variational problem (Lagrangian and the boundary term) for which this equation is the Euler equation.

5. Find an approximate solution to the problem

$$I(u) = \min_{u(x)} \int_0^1 \left(x(u'-1)^2 + 3x^2u^2 \right) dx \; u(0) = u(1) = 0$$

assuming a representation $u(x) = x(1-x)(A_0 + A_1 x + A_2 x^2)$. Find the constants A_0, A_1, A_2. You may use Maple or similar software.

6. Show that the Euler equation of the problem

$$\inf_{u(x)} \int_0^1 \left((u-x)^2 + \min\left\{ (u'-1)^2, \; (u'+1)^2 \right\} \right) dx, \quad u(0) = 0$$

does not satisfy the Weierstass test. Describe a minimizing sequence. Find the relaxed formulation of the problem.

7. Find Euler-Lagrange equation and natural boundary conditions for the problem

$$I(n) = \min_{u(x)} \left[\int_\Omega \left((\nabla u)^2 - q\, u \right) dx + \oint_\Gamma \left((c\, u + d\, u^2 \right) ds \right], \quad \int_\Omega u^2 \, dx = 1$$

where $x = (x_1, x_2)$, Ω is a bounded domain in R_2 with the smooth boundary Γ, $u(x)$ is a scalar minimizer, $q, \; c, \; d$ are parameters.

Extra-Credit Problem

Show that the Euler equation of the problem

$$\inf_{u(x)} \int_0^1 \left((u-\alpha x)^2 + \min\left\{ (u'-1)^2, \; (u'+1)^2 \right\} \right) dx, \quad u(0) = 0$$

does not satisfy the Weierstass test for certain values of α. Find these values. Describe a minimizing sequence. Find the relaxed formulation of the problem.