HW 4. Numerical solutions

1. Approximation methods
Approximate solution and the cost of the problems:

$$I_1 = \min_{u(x0,\ u(0)=0, u(\pi)=0} \int_0^\pi \sin(\pi x)(u')^2 + (u)^2 dx \text{ if } \int_0^\pi u\dot{d}x = 1$$

and

$$I_2 = \min_{u(x0,\ u(0)=0, u(\pi)=0} \frac{\int_0^\pi x(u')^2 dx}{\int_0^\pi (1-x)(u)^2 dx}$$

representing $u(x)$ as a polynomial or trigonometric polynomial of the second order and finding the coefficients.

To check, apply same procedure to the problem

$$\min_{u(x0,\ u(0)=0, u(\pi)=0} \frac{\int_0^\pi (u')^2 dx}{\int_0^\pi (u)^2 dx}$$

with the known answer.

2. Gradient method
Let $u(x)$ be the solution of brachistochrone problem

$$\min u(x) \int_0^1 F(u, u')dx, \quad F = \frac{\sqrt{1+u'^2}}{\sqrt{2g\,u}}$$

and $u(0) = a$ and $u(1) = b$.

Compute the first iteration $u_1(x)$ of U(x) using representation

$$u_0(x) = (b-a)x + a, \quad u_1(x) = u_0(x) - \epsilon x\,(1-x)S(u_0)$$

where $S(u)$ is the functional derivative:

$$S(u) = \frac{\partial F}{\partial u} - \frac{d}{d\,x}\frac{\partial F}{\partial u'}$$

Notice that multipliers $x\,(1-x)$ fix the boundary conditions. Compare with the known solution choosing a and b. Try different values of ϵ.

Compute the next iteration.