HW 8. Multiple integrals

1. Find Euler-Lagrange equations and the natural boundary conditions:

$$\min_u \int_\Omega \Big((\nabla u_1)^2 + (\nabla u_2)^2 + a \det(\nabla u_1 | \nabla u_2) - u_1 u_2 \Big)\, dx.$$

Does the solution depend on a?

2. Find the Lagrangian if the Euler-Lagrange equation is

$$\nabla \cdot (\sigma(x) \nabla u) + \phi(x) = 0 \ \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + \gamma u = 0 \text{on } \partial\Omega.$$

3. Consider the problem

$$\min_u \int_\Omega \Big(a\, u^2 + b (\nabla \times u)^2 \Big)\, dx.$$

1.Derive the Euler-Lagrange equations and natural boundary conditions.
2. Rewrite the problem as

$$\min_u \int_\Omega \Big(a\, u^2 + b\, v^2 \Big)\, dx \text{ subject to } v = \nabla \times u,$$

use Lagrange multiplier to account to the last constraint, derive the Euler-Lagrange equations and the Euler-Lagrange equations for a dual problem (exclude u).

4. Derive conditions for an optimal boundary component for a problem

$$\min_\Omega \min_u \int_\Omega \left((\nabla u)^2 + \gamma \oint_{\partial\Omega} u^2 \right) ds \quad \|\Omega\| = R$$