Final Exam
M 5500, Spring 2017

Solve any four problems

1. Find Euler equation and natural boundary condition, check Legendre condition
$$I(u) = \min_{u(x)} \int_0^1 \left((u')^2 + 3\sin(x)u^2 \right) dx \quad \text{if} \quad \int_0^1 (u)\, dx = 1$$

2. Two equal masses $m = 1$ with coordinates $r_1(t)$ and $r_2(t)$ move in a plane, $(r_i(t) = (x_i(t), y_i(t))$ minimizing the difference of kinetic and potential energy
$$V = \frac{\gamma}{|r_1 - r_2|}$$
The initial conditions of the masses are given.

   Find Lagrangian and Hamiltonian, equations of motion of the masses. Find invariants to the motion. Hint: Express coordinates through the center of masses $r_0 = \frac{r_1 + r_2}{2}$.

3. Find an approximate solution to the problem
$$I(u) = \min_{u(x)} \int_0^1 \left(x(u' - 1)^2 + 3x^2u^2 \right) dx \ u(0) = u(1) = 0$$
assuming a representation $u(x) = (1 - x)(A_0 + A_1x + A_2x^2)$. Find the constants A_0, A_1, A_2.

4. Find Euler-Lagrange equation for the problem
$$I(n) = \min_{u(x)} \left[\int_\Omega \left((\nabla u)^2 - q(x)u \right) dx + \oint_{\partial\Omega} \left(c(x)u + d(x)u^2 \right) ds \right],$$
where $x = (x_1, x_2)$, $u(x)$ is a scalar minimizer, $q(x), c(x), d(x)$ are given functions.

5. Minimize the length of the boundary between two domains of given areas A_1 and A_2. Sketch the optimal shapes.