Contents

Chapter 7

Variation of the domain

Μὴ εἶναι βασιλικὴν ἀτραπὸν ἐπὶ γεωμετρὶαν [Εὐκλειδην]
There is no royal road to geometry [Euclid]

7.1 Variation of the domains: Setting

Here, we derive the necessary conditions of optimality for the domain Ω where the Lagrangian in defined. The one-dimensional analog of this problem is the variation of the interval that leads to the transversality conditions, see Section (**??**). First, we derive optimality condition for the isoperimetric and related problem, where the Lagrangian is degenerative: it depends solemnly on the domain (like volume, inertia moment, or perimeter). Then, we consider a general problem for Lagrangian depending on a minimizer inside the domain.

Consider a region $\mathcal{O}$ ($\mathcal{O}$ can coincide with R_2) and a set of admissible domains Ω with the twice differentiable boundaryγ that belongs to the interior of $\mathcal{O}$. Consider also a differentiable minimizer $u(x)$, $x \in \mathcal{O}$, and a twice differentiable Lagrangian $F(x, u, \nabla u)$. Consider the variational problem (Mayer-Bolza problem).

$$J(\Omega) = \min_{u(x)} \left\{ \int_\Omega F(x, u, \nabla u) dx + \int_\gamma f(x, u) ds \right\}, \quad x \in \Omega \tag{7.1}$$

that contains the bulk and the boundary integrals. The objective functional $J(\Omega)$ depends on the domain Ω.

Consider domain optimization: Find Ω that minimizes $J(\Omega)$ We come to a variational problem for the variable domain

$$I = \min_{\Omega \subset \mathcal{O}} J(\Omega). \tag{7.2}$$

Here, we study stationarity condition for optimal Ω. In order to define variation of the domain, assume that functions $F(x, u, \nabla u)$ and $f(x, u)$ are defined everywhere in a larger region $\mathcal{O}$. For simplicity, we first discuss the two-dimensional problem.

Figure 7.1: Infinitesimal variation of the boundary of the domain

Variation of the boundary We describe the boundary variation $\delta\gamma$ and related variation $\delta\Omega$. Consider an admissible domain $\Omega \cup \delta\Omega$ that differs from Ω by an infinitely thin domain $\delta\Omega$ with a twice differentiable boundary $\gamma \cup \Gamma$, as on Figure 7.1. Denote the added boundary of this domain by $\Gamma(s)$. In other words, we consider the added/subtracted domain $\delta\Omega$ as an infinitesimally thin strip of the width $\delta\eta$ and the boundary $\gamma \cup \Gamma$. Asusual, we compute the variation of the objective

$$\delta I = J(\Omega \cup \delta\Omega) - J(\Omega)$$

and analyze the stationary condition $\delta I = 0$ which provide an additional boundary condition at the unknown optimal boundary γ. This way, we formulate the *free boundary problem*: a boundary value problem in an unknown domain but with an additional condition for this domain.

Remark 7.1.1 The considered variation does not change the topology of the domain: No new components of the boundary were added.

We start with establishing the coordinate system in proximity of $\partial\Omega$, particularly, the correspondence between points of the stationary boundary γ and the varied close-by boundary Γ. Define normal $n = n(s)$ to each point γ; the normal of a twice-di?erentiable curve is a continuous function of s. Introduce the following coordinates in an infinitisimal proximity of γ: a distance s along the curve and the (infinitesimal) distance η along the normal $n(s)$. Particularly, each point of γ is represented by $(s, 0)$. The admissible close-by curve Γ is represented by infinitesimal distance $\delta\eta$, $\Gamma: \ \{s, \delta\eta(s)\}$. Notice that $\delta\eta(s)$ may have any sign, the variation can go inside or outside of Ω.

A point in an infinitesimal strip $\delta\Omega$ can be defined by two coordinates: The distance s along the boundary line γ and the distance $z\delta\eta$ from this line computed along the normal, where $z \in [0, 1]$. Particularly, points at δ correspond to $z = 0$ and points at Γ – to $z = 1$.

We consider "thin and small" variations assuming that

$$\delta\eta = 0 \ \text{ if } s \notin (s_0,\, s_0 + \epsilon), \quad \delta\eta \ll \epsilon \tag{7.3}$$

If the curvature k of γ is not zero, the length of the arch varies with the normal. The infinitesimal length dS along Γ is related to the infinitesimal

Figure 7.2: The variation of the differential of a curve due to its curvature

length ds along γ as

$$dS = ds(1 + k\delta\eta) + o(\delta\eta) \tag{7.4}$$

This formula is known in differential geometry as ???? , see [] (see for example **??**) Figure 7.2 illustrates it:

Remark 7.1.2 Here, we assume that Ω belongs to an open set $\mathcal{O}$. If this set is closed, the consideration must be supplemented by variational inequalities (see Section **??**). Additional constraints on Ω such as the prescription of its volume or perimeter can be conventionally accounted by the Lagrange multipliers.

Remark 7.1.3 The variation of singular points of the domain where the normal is not defined must be considered separately. Such variations will be shown on the examples below.

7.2 Geometric Lagrangian

7.2.1 Stationarity conditions

First we show how the bulk and boundary integrals in (7.1) vary due to the variation of the boundary in the case when the integrands are independent of u and are continuous bounded functions of x only, $F = F(x)$ and $f = f(x)$. In other words, consider the variational problem

$$I = \min_{\Omega \subset \mathcal{O}} J(\Omega), \quad J(\Omega) = \int_\Omega F(x)dx + \int_\gamma f(x)ds, \quad x \in \Omega. \tag{7.5}$$

that depends only on the shape of the region Ω.

Variation of the integral over Ω The integral of a continuous function $F(x)$ over an infinitely thin domain is estimated through the boundary integral

$$I_{\delta\Omega} = \int_{\delta\Omega} F(x)dx = \int_\gamma F(s)\delta\eta(s)ds + o\|\delta\eta(s)\|. \tag{7.6}$$

To obtain this formula, we compute the integral $I_{\delta\Omega}$ as the repeated integral

$$I_{\delta\Omega} = \int_\gamma \delta\eta \left(\int_0^1 F(s, \delta\eta\, z)dz \right) ds \tag{7.7}$$

integrating first along the normal n and then over the arch γ. The inner integral is estimated using the continuity of F and the smallness of $\delta\eta$, as

$$\int_0^1 F(s, \delta\eta z)dz = \delta\eta\, F(s, 0) + o(\delta\eta)$$

and the formula (7.6) is obtained by computing the first-order term. Here, we denote $F(s, 0)$ as $F(s)$.

Variation of the boundary integral The value of the differentiable function $f(x)$ at the point $S \in \Gamma$ is expressed as

$$f(S) = f(s) + \delta\eta \frac{\partial f}{\partial n} + o(|\delta\eta|) \tag{7.8}$$

where $\frac{\partial f}{\partial n} = n^T \nabla f$ is the normal derivative of f.

The variation of an integral of a function $f(x)$ can be computed using (7.4) and (7.8) and rounding to $|\delta\eta|$

$$\begin{aligned} \int_\Gamma f(S)dS - \int_\gamma f(s)ds &= \int_\gamma \left[\left(f(s) + +\delta\eta \frac{\partial f}{\partial n} \right) (1 + k\delta\eta) - f(s)) \right] ds \\ &= \int_\gamma \left(kf(s) + \frac{\partial f(s)}{\partial n} \right) \delta\eta \, ds. \end{aligned} \tag{7.9}$$

Stationarity conditions Adding together the two increments (7.6) and (7.9), we find that the increment δI in (7.5) is equal to

$$\delta I = \int_\gamma \left(F(s) + kf(s) + \frac{\partial f(s)}{\partial n} \right) \delta\eta \, ds.$$

As usual, we use the arbitrariness of $\delta\eta$ and obtain the stationary condition

$$F(s) + \left(k + \frac{\partial}{\partial n} \right) f(s) = 0 \quad \forall s \in \gamma \tag{7.10}$$

for this simplest case (7.5) where the Lagrangian is a fixed function of x.

Example 7.2.1 (Isoperimetric problem) Consider the isoperimetric problem: Maximize the area of Ω keeping its perimeter equal to one:

$$\max_\Omega \int_\Omega dx \quad \text{subject to} \quad \int_\gamma ds = 1 \tag{7.11}$$

or

$$\max_\Omega \left\{ \int_\Omega dx + \Lambda \int_{\partial\Omega} ds \right\}$$

where Λ is the Lagrange multiplier by the isoperimetric constraint. Here we put

$$F = 1, \quad f = \Lambda$$

The necessary condition (7.10) gives $1 + k\Lambda = 0$

$$k = -\frac{1}{\Lambda} = \text{constant} \tag{7.12}$$

at the optimal boundary. The unknown boundary is a circle (or a part of a circle).

Remark 7.2.1 (Comparison with symmetrization technique) The isoperimetric problem can be solved by the symmetrization method, see Sectionsymm. Contrary to the symmetrization method, the variational technique is applicable to a wider range of problems. Here, for instance, the necessary conditions are applicable for the case if a part of the boundary of Ω is fixed. The unfixed boundary components are the arcs of constant curvature, joined by the fixed boundary components. The curvature of the unfixed components is constant everywhere and can be found from the isoperimetric condition.

On the other hand, the obtained condition do not prove that the shape with constant curvature deliver the global minimum of the isoperimetric problem because only close-by trajectories were compared, while the symmetrization method guarantees the global minimum but cannot withstand these additional constraints.

Example 7.2.2 (A domain with an extremal moment and fixed perimeter) Consider the problem of the symmetric domain with maximal moment of inertia M

$$M = \int_\Omega x^2 dx \tag{7.13}$$

that passes through two given points $(-a, 0)$ and $(a, 0)$ and has a fixed perimeter. We set $F = x^2$, $f = \lambda$, where λ is the Lagrange multiplier. The optimality condition is $\lambda k = x^2$. In a Cartesian coordinates, the boundary of the domain consists of a pair of curves $\pm y(x)$, the curvature is given by the known expression $k = \frac{y''}{1+y'^2}^{\frac{3}{2}}$, and the optimality condition leads to the equation

$$y'' = \frac{1}{\lambda} x^2 (1 + y'^2)^{\frac{3}{2}}$$

Separating the variables and using symmetry ($y'(0) = 0$), we obtain

$$y'(x) = \frac{x^3 \lambda}{\sqrt{9 - x^6 \lambda^2}}$$

Next , we find $y(x)$ and calculate λ using condition (7.13). The graph of the optimal curve is shown at Figure **??**.

Notice that the curve is smooth everywhere except the points $(-a, 0)$ and $(a, 0)$ where its values are prescribed.

7.2.2 Geometric applications

7.2.3 Cluster of domains with minimal boundary length

Boundary components Assume that two finite domains Ω_1 and Ω_2 of given areas A_1 and A_2, respectively, have a common component γ_{12} of the boundary. Consider the problem of the shapes of these domain that minimizes the total length of the boundary.

Figure 7.3: Optimal angle between normals in the singular point
Left: The angle between any two normal is less than 180°
Right: The angle between any two normal equals 120°

The domain $\Omega = \Omega_1 \cup \ldots \Omega_N$ has outer and inner boundaries. The optimality conditions for variation of the outer boundary is similar to the condition (7.12) in the isoperimetric problem:

$$\delta\eta_i : \quad k_i + \Lambda_i = 0, \quad i = 1, 2.$$

The conditions indicate that the outer bound is composed of two circular arcs with radii $R_1 = 1/k_1$ and $R_2 = 1/k_2$, respectively.

The variation of an inner component γ_{12} of the boundary γ_{12} increases the volume of one of the neighbor and decreases the volume of the second one by the same amount. The resulting condition is:

$$\delta\eta_{12} : \quad k_{12} + \Lambda_1 - \Lambda_2 = 0$$

This implies that the dividing lines are also circular, with the radius (a reciprocal to the curvature) R_{ij} equal to

$$\frac{1}{R_{12}} = \frac{1}{R_1} - \frac{1}{R_2} \tag{7.14}$$

In particular, the boundary between two domains of equal areas is straight. Smaller domain remains convex when it is joined with the larger one, and the larger is of crescent shape.

Singular points To complete the consideration, we determine the angle at the point s_0 where three components of the boundary meet. This problem requires special consideration because the boundary curve is not smooth, the normal at the point s_0 is not defined, so the standard variations are not possible.

First, it is easy to show that the components stay continuously differentiable until they meet. Assume that a boundary component has an angular point s_0 and the normal is discontinuous there, denote the angle ϕ. Consider two points infinitely close points s_1 and s_2 at the boundary at different sides of s_0 such that $|s_1 - s_0| = |s_2 - s_0| = \epsilon$ and a triangle $s_0 s_1 s_2$. Assume that the varied boundary replaces curve $s_2 s_0 s_1$ with the straight line $s_1 s_2$ of the length $|s_1 s_2| = 2\epsilon \cos\phi$. The variation δl of boundary lengths is negative $|Gdl = 2$ (which is evident from triangle inequality) and is of the order ϵ, the variation of the area $\delta A = \epsilon^2(\cos\phi - 1)$ is neglected in linear part of increment. We conclude that there cannot be kinks at an opptimal boundary.

Consider now a meeting point s_0 of three boundary components. Again, choose three points s_1, s_2 and s_3 at the corresponding branches of the boundaries in ϵ-neighborhood of s_0, so that $s_i \neq s_0$. s_0 is optimal, if

$$L(s_0) = \sum_{i=1}^{3} |s_i - s_0|$$

is minimal. Differentiation with respect to s_0, we obtain equation for an optimal s_0,

$$\frac{dL(s_0)}{ds_0} = \sum_{i=1}^{3} \frac{s_i - s_0}{|s_i - s_0|} = 0.$$

If the sum of three unit vectors is zero, these vectors are directed at 120° to each other.

Remark 7.2.2 The minimization problem also has another solution, $s_0 = s_2$ that is optimal when the angle between $(s_1 - s_2)$ and $(s_2 - s_3)$ is less than a threshold (see Problem **??**). Here, however, we assumed that the branch point is not equal to any of s_1, s_2 and s_3, for optimal contours.

Again, the increment of the variation of lengths is of the order of ϵ and the increment of areas is of the order of ϵ^2 and is neglected in linear term. We conclude that

Theorem 7.2.1 Optimal boundary components meet at the angle 120° to each other.

Definition of parameters We found that the boundaries of two-domain configuration are consist of three arches of circles which centers lie on a line and radii are connected by the condition (7.14); that intersect in two symmetric points, the angle between the arches equal to 120°. To draw the configuration, it is enough to find radii R_1, R_2 and R_{12}, and angles θ_1, θ_2 of the arches. To find these five parameters we have two equations that express the conditions of intersections

$$R_1 \sin\theta_1 = R_2 \sin\theta_2 = R_{12} \sin\theta_{12}$$

equation (7.14), and two equations that fix the areas of the domains

$$A_1 = R_1(\pi - \theta_1) + \frac{1}{2} R_1^2 \sin\theta_1 \cos\theta_1 + R_{12}\theta_{12} - \frac{1}{2} R_{12}^2 \sin\theta_{12} \cos\theta_{12}$$

and

$$A_2 = R_2(\pi - \theta_2) + \frac{1}{2} R_2^2 \sin\theta_2 \cos\theta_2 - R_{12}\theta_{12} + \frac{1}{2} R_{12}^2 \sin\theta_{12} \cos\theta_{12}$$

and the condition

$$\theta_1 - \theta_{12} = \theta_2 - \theta_{12} = 2\pi/3$$

too many conditions!

Summing up, the formulate the theorem:

Theorem 7.2.2 The optimal boundary consists of three circular arches that meet at two symmetric points at $120°$; the curvatures of the arches are related as $k_{12} = k_1 - k_2$.

For instance, if the areas are very distinct, $A_1 \ll A_2$, the smaller area is a lens made of the circles of the close-by radii circle, and the larger area is a circle without a lens-shaped area.

Multiple domains A natural generalization of the previous problem is the problem of N separated domains Ω_i, $i = 1, \dots, N$ of given areas A_i with the minimal length of separating boundary. The domain $\Omega = \Omega_1 \cup \dots \Omega_N$ has outer and inner boundaries. The optimality conditions for variation of the outer boundary are similar to the condition (7.23) in the isoperimetric problem:

$$\delta\eta_i : \quad k_i + \Lambda_i = 0$$

Let us find the conditions of optimality of an inner component of the boundary. Suppose that two domains Ω_i and Ω_j are neighbors. The variation of their common boundary ∂_{ij} results in the condition

$$\delta\eta_{ij} : \quad k_{ij} + \Lambda_i - \Lambda_j = 0$$

This implies that the dividing lines are also circular, with the radius (a reciprocal to the curvature) R_{ij} equal to

$$\frac{1}{R_{ij}} = \frac{1}{R_i} - \frac{1}{R_j}$$

If the number of domains with different areas is larger of equal to four, the question arise what domains should be placed inside the configuration and do not have an outer boundary. More generally, the question is which domains should be neighbors. This problem requires combinatoric methods since it is needed to compare several not close-by configurations. We do not discuss the problem in this text but encourage the reader to try to solve it.

Example 7.2.3 An infinite system of domains of equal areas forms a honeycomb structure. Bees know variational calculus! Indeed, all the inner boundaries are straight due to symmetry, and all angles where the boundaries meet are equal to $120°$.

Optimal shapes in a bounded domain When a shape touches the boundary of the domain, it meets the this boundary at a right angle. Indeed, the angle which correspond to minimal length of γ in the to the boundary is a right angle. The rest of the consideration is as before.

Natural cracks in Badlands Photo! Observe the cracks meeting either at 120^o or at 90^o Discuss the history of development of the cracks.

7.2.4 Three-dimensional problems: Minimal surface and shape of bubbles

Three-dimensional problem The consideration of the three-dimensional case is quit the same, but the formula for the variation of the boundary arch is replaced by the formula for the variation of the boundary surface element:

$$dS = ds(1 + k_1\delta\eta)(1 + k_2\delta\eta) = ds(1 + (k_1 + k_2)\delta\eta) = o(\delta\eta) \tag{7.15}$$

where k_1 and k_2 are the main curvatures on the boundary surface. The calculation of the stationary condition is performed as before. The necessary condition:

$$F(x) + \left(k_1 + k_2 + \frac{\partial}{\partial n}\right) f = 0 \tag{7.16}$$

differs from the two-dimensional case by replacing the curvature k of the boundary line with the mean curvature $k_1 + k_2$ of the boundary surface.

The simplest problem is the minimal surface problem: Find a surface of minimal area that is attached to a given contour. Here,

$$F = 0 \quad f = 1$$

Remark 7.2.3 The surface may be not closed, but this is of no importance since we consider only local variation of the "boundary" surface.

The stationarity condition

$$k_1 + k_2 = 0$$

shows that the minimal surface has zero mean curvature.

Bubble The bubble problem is the three-dimensional analog of the isoperimetric problem: Find a domain of maximal volume enclosed in the surface of fixed area. Here

$$F = 1 \quad f = \Lambda$$

and the optimality condition

$$k_1 + k_2 = \Lambda$$

stays that the mean curvature is constant. A sphere is an obvious solution. Beside it there are many other shapes: a circular cylinder, for example. One may experiment with an air-balloon to find these shapes. The problem is formulated as a partial differential equation that expressed the constancy of the mean curvature of the surface with the boundary conditions that correspond to the requirements that the surface passes through a given contour. For references, see [].

Cluster of bubbles Similarly to the two-dimensional case, a variational problem of minimization the total area around several spacial domains with given volumes is considered. Physically, this is the problem of a cluster of bubbles.

The cases of two, three, and four equal bubbles can be handled out analytically using the obvious symmetries. In the general case, the outer and interior surfaces of the cluster satisfy the following conditions:

Theorem 7.2.3 (Bubble theorem) The following statements describe the cluster of the bubbles:

(1) The outer boundary of the cluster consists of the surface with piece-wise constant mean curvature.

(2) Any two volumes (bubbles) are divided by a surface of constant mean curvature.

(3) Any three bubbles are divided by a curves formed by intersection of three mean curvature surfaces with normals that meet at the angles $2\pi/3$ independently of the volumes of bubbles.

(4) Any four bubbles meet at the points where four mean curvature surfaces meet. The normals to these surfaces at the meeting point meet at the angles $2\pi/3$ as the planes meet in the center of a symmetric tetragon that pass through the sides of this tetragon. The angles between the meeting planes are fixed independently of the volumes of bubbles.

Find a reference

Hints for the proof
(1) Apply the necessary conditions on a free boundary
(2) Apply the necessary conditions on a dividing boundary
(3) Consider an infinitesimally thin cylinder abound the line of intersection.
(4) Consider an infinitesimal sphere or tetragon around the point of intersection.

7.3 General type Lagrangian

7.3.1 Formulas for the increment

Here we consider the general form of the functional (7.2), assuming that that the integrants depend on a minimizer $u(x)$, $x \in \Omega$ and the domain Ω is varied. We derive an additional boundary condition at the optiaml domain which serves for finding this domain.

The variation of the functional in (7.2) is computed as the difference

$$\begin{aligned} \delta J_n &= \delta I_1 + \delta I_2 \\ \delta I_1 &= \int_{\Omega\cup\delta\Omega} F(x, u+\delta u, \nabla(u+\delta u))dx - \int_\Omega F(x,u,\nabla u)dx \\ \delta I_2 &= \int_\Gamma f(x, u+\delta u)dS - \int_\gamma f(x,u)ds \end{aligned} \tag{7.17}$$

between the cost of the Lagrangians of an admissible solution in an admissible domain and of the extremal solution in the optimal domain.

Variation of the bulk integral First, we work out the increment δI_1 of bulk integrals. We rewrite it as

$$\delta I_1 = \int_{\Omega \cup \delta\Omega} F_u \delta u \, dx + \int_{\delta\Omega} F(x, u, \nabla u) dx + \int_{\Gamma} n \cdot \frac{\partial F}{\partial \nabla u} \delta u(\Gamma) \, dS + o(\|\delta u\|, \|\delta\eta\|) \tag{7.18}$$

where refer to formula instead

$$F_u = -\nabla \cdot \frac{\partial F}{\partial \nabla u} + \frac{\partial F}{\partial u}$$

To derive this formula, we add and subtract the integral over $\delta\Omega$ of $F(x, u, \nabla u)$ and compute the fist variation δI_1 with respect to δu in $\Omega \cup \delta\Omega$.

The variation δu at $\Gamma = \gamma + \delta\eta$ that is a sum of the variation of minimizer $\delta u(\gamma)$ at stationary boundary γ and shift $\frac{\partial F}{\partial u} \delta\eta$ of the minimizer due to variation of the boundary,

$$\delta u(\Gamma) = \delta u(\gamma) + \frac{\partial u}{\partial n} \delta\eta$$

In other words, variation $\delta u(\gamma)$ is expressed as the linear combination of two free variations $\delta u(\Gamma)$ and $\delta\eta$:

$$\delta u(\gamma) = \delta u(\Gamma) - \frac{\partial u}{\partial n} \delta\eta$$

Integral over $\partial\Omega$ in the right-hand side of (7.22) is estimated as

$$\int_{\Omega \cup \delta\Omega} F(x, u, \nabla u) dx = \int_{\gamma} F(x, u, \nabla u) \delta\eta dx + o(\|\delta\eta\|)$$

It remains to substitute these expressions into (7.22) and group the terms:

$$\delta I_1 = \int_{\Omega} F_u(x, u, \nabla u) \delta u \, dx + \int_{\gamma} A_u \delta u \, ds + \int_{\gamma} A_\eta \delta\eta \, ds$$

$$A_u = n \cdot \frac{\partial F}{\partial \nabla u} \quad A_\eta = F - \left(n \cdot \frac{\partial F}{\partial \nabla u} \right) \frac{\partial u}{\partial n} \tag{7.19}$$

The first two terms are standart conditions of stationarity of u, the last term expressed the stationarity of Ω. Increment A_η is analogous to the transversality condition (7.23) in the one-dimensional problem. It is the sum of the accumulated value of the Lagrangian over the added domain $\delta\Omega$ and (term $-n^T \frac{\partial F}{\partial \nabla u} \delta\eta$) a shift of natural boundary conditions from the admissible to the stationary boundary. A_η is an analog of the term $-u' \frac{\partial F}{\partial u'} \delta x$ in the one-dimensional transversality condition.

If the boundary integral is zero, the additional (transversality) boundary condition $A_\eta = 0$ holds at the unknown boundary γ and serves to find this boundary.

First variation of the boundary integral To compute the curve integral I_2, we expand f as follows:

$$f(X, u(X))|_{X\in\Gamma} = f(x)|_{x\in\gamma} + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial n}\right)\delta\eta + \frac{\partial f}{\partial u}\delta u + o(\|\eta\|)$$

and take into account the variation of the boundary arch (7.4) ($dS = ds(1 + k\delta\eta)$), obtaining:

$$\delta I_2 = \int_\gamma \left[f(x,u) + \frac{\partial f}{\partial u}\delta u + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial n}\right)\delta\eta\right](1 + k\delta\eta)ds - \int_\gamma f(x,u)ds$$

Rounding to $\delta\eta$, we have

$$\delta I_2 = \int_\Gamma \left[\left(k\, f + \frac{\partial f}{\partial n} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial n}\right)\delta\eta + \frac{\partial f}{\partial u}\delta u\right] ds$$

The first terms in parenthesis accounts to increase of the length of varied curve Γ, the next two terms show the shift of the $f(s)$ due to variation of s, and the last term is the variation due to variation of u.

Stationary conditions Adding δI_1 and δI_2, we finally obtain the stationarity conditions:

The variations with respect to u returns the familiar expressions: F_u is the left-hand-side of the Euler equation

$$\delta u(x): \quad F_u = \nabla \cdot \frac{\partial F}{\partial \nabla u} - \frac{\partial F}{\partial u} = 0 \quad \text{in } \Omega,$$

natural boundary condition,

$$\delta u(s): \quad \Phi_u \delta u(s) = 0, \quad \Phi_u = \frac{\partial F}{\partial \nabla u}\boldsymbol{n} - \frac{\partial f}{\partial u} = 0 \quad \text{on } \gamma, \tag{7.20}$$

and the stationarity of the domain itself:

$$\delta\eta: \quad F(x,u,\nabla u) + \left[-\frac{\partial F}{\partial \nabla u}\cdot n + \frac{\partial f}{\partial u}\right]\frac{\partial u}{\partial n} + \frac{\partial f}{\partial n} + k\, f(x,u) = 0.$$

If no boundary condition is prescribed (free boundary, or variational boundary conditions), then $\Phi_u = 0$ due to (7.20), and the last condition is simplified to

$$R_F = F(x,u,\nabla u) + \frac{\partial f}{\partial n} + k\, f(x,u) = 0. \tag{7.21}$$

It serves to determine an unknown boundary with variational boundary conditions.

Remark 7.3.1 The last condition involves the dependence of ∇u which we represent in local coordinates as $\nabla u = \frac{\partial u}{\partial n}\boldsymbol{n} + \frac{\partial u}{\partial s}\boldsymbol{t}$; generally, it leads to a differential equation

$$\Psi\left(\frac{\partial u(s)}{\partial s}, u(s), \frac{\partial u(s)}{\partial n}\right) = 0$$

along the boundary. In this equation, $u(s)$ and $\frac{\partial u(s)}{\partial n}$ are independent functions. To define both of them, we consider also the natural boundary condition (7.20) that expresses another relation $\psi\left(u(s), \frac{\partial u(s)}{\partial n}\right) = 0$ between these functions.

Example 7.3.1 (Vibrating membrane of minimal frequency) Consider the problem of the vibration membrane of a fixed area with minimal first frequency F in the form $F = \frac{1}{2}\nabla^2 u - \frac{1}{2}c^2u^2 + \Lambda$ where Λ is Lagrange multiplier by the condition fixing the area, u is the deflection, and ω i the frequency. on the boundary γ, the membrane is fixed, $u = 0$. In this problem, $f = 0$, and $\frac{\partial u}{\partial s} = 0$ on γ.

It is convenient to compute R_F in local coordinates – the tangent t and the normal n to the boundary. In this coordinates, we compute

$$F = \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial n}\right)^2 - \frac{1}{2}c^2u^2 + \Lambda = \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \Lambda.$$

We have

$$R_F = -\frac{1}{2}\left(\frac{\partial u}{\partial n}\right)^2 + \Lambda = 0 \quad \text{or} \quad \frac{\partial u}{\partial n} = \text{ constant} \quad \text{on } \gamma$$

and it serves to find the unknown boundary γ. One easily guesses that the circular membrane has an optimal shape.

Three-dimensional problem In the three-dimensional case, the formula for the variation becomes:

$$F(x, u, \nabla u) + \frac{\partial f}{\partial n} + (k_1 + k_2)f = 0$$

Here, we use formula (7.23) for the variation of boundary element.

Conditions on the free boundary Assume that the natural boundary conditions (7.23) are imposed on the unknown component of the boundary. In many physical applications, $F = F(u, \nabla u)$ means the bulk energy, and $f = f(u)$ is the surface energy. We also assume that both forms of energy do not explicitly depend on the position x,

$$F = F(u, \nabla u), \quad f = f(u)$$

Due to the listed conditions, the optimality condition on the unknown moving boundary simplifies and becomes:

$$F(u, \nabla u) - (k_1 + k_2)f(u) = 0 \tag{7.22}$$

We see that the mean curvature of the optimal boundary surface is the proportionality coefficient between the bulk and surface energies.

In particular, the constant value of f corresponds to the prescription of the surface area. In this case, the bulk energy density on the boundary is proportional to its mean curvature.

Inner boundary Similar consideration is applicable to the problem of optimal position of a variable boundary γ dividing two volumes Ω_1 and Ω_2. The problem is to minimize the energy

$$\int_{\Omega_1} F_1(u, \nabla u)dx + \int_{\Omega_2} F_2(u, \nabla u)dx + \int_\gamma f(u)dx \tag{7.23}$$

The necessary conditions for the boundary γ are

$$n^T \frac{\partial}{\partial \nabla u}[F(u, \nabla u)]_-^+ + \frac{\partial f(u)}{\partial u} = 0 \quad \text{on } \gamma$$

(the natural boundary condition on any boundary between the volumes, obtained by the variation of u), and

$$[F(u, \nabla u)]_-^+ + (k_1 + k_2) f(u) = 0 \quad \text{on } \gamma$$

Here, $[F]_-^+ = F_2 - F_1$ denotes the jump: The difference in the values of the function F at both sides of γ. Notice that the solution u is continuous at the boundary, and so is its tangential derivative $\frac{\partial u}{\partial t}$ but the normal derivative $\frac{\partial u}{\partial n}$ is, generally, discontinuous.

7.3.2 Application: Optimal Shape of Conducting Domains

Next problems ask for the shape of the domain of extremal resistivity. Consider a domain Ω with the boundary $\gamma = \gamma_0 \cup \gamma_1 \cup \gamma_i$ divided into three components. Assume that the domain is filled with an isotropic material with unit conductivity. Assume that the potentials $u = 0$ and $u = 1$ are applied to the two components γ_0 and γ_1 of the boundary, respectively. The supplementary component γ_i of the boundary is insolated,

$$u = 0 \quad \text{on } \gamma_0, \quad u = 1 \quad \text{on } \gamma_1, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \gamma_i, \tag{7.24}$$

The resistivity is defined as the total conducting energy of the domain. Indeed, the total normal current $j = \frac{\partial u}{\partial n}$ through the γ_1 component of the domain is equal to the total energy of it:

$$\int_{\gamma_1} \frac{\partial u}{\partial n} ds = \int_\gamma u \frac{\partial u}{\partial n} ds = \min_u \frac{1}{2} \int_\Omega (\nabla u)^2 \, dx$$

The first equality follows from the boundary conditions (7.24) and the second – from Green's formula combined with the stationary condition $\nabla^2 u = 0$ in Ω.

Let us formulate an optimization problem: Maximize the total current through the γ_1 or, equivalently, minimize the total conductivity of the domain by varying its boundary. Additional geometrical restriction can be assigned.

Assume in addition that the area s of the plain domain Ω is fixed. Consider the conductivity problem in a domain of the fixed volume and a partly known boundary

$$\min_{\gamma_i} \left(\min_{u \text{ as in (7.24)}} \frac{1}{2} \int_\Omega (\nabla w)^2 dx + \lambda \int_\Omega dx \right)$$

where λ is the Lagrange multiplier. The potential u on the known components is prescribed and natural boundary conditions are satisfied on γ_i.

The augmented Lagrangian is

$$F = (\nabla u)^2 + \lambda, \quad f = 0$$

and λ is the Lagrange multiplier. The Euler-Lagrange equation and natural boundary condition are

$$\nabla^2 u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \gamma_i$$

One more condition is needed to define the shape of unknown boundary component γ. This condition on the unknown boundary is

$$n^T \frac{\partial F}{\partial \nabla u} - F = \left(\frac{\partial u}{\partial n}\right)^2 - \left(\frac{\partial u}{\partial t}\right)^2 - \lambda = 0 \tag{7.25}$$

where t is the tangent to the boundary.

Example 7.3.2 (Conditions on optimal insolated boundary component) Assume that the shape of boundary components γ_0 and γ_1 is known, but the insolated component γ_i is movable. Combining the optimality condition (7.25) with the stationary condition $\frac{\partial u}{\partial n} = 0$ on γ_i we obtain the condition for the unknown boundary:

$$w_t = \sqrt{-\lambda} = \text{constant} \quad \text{on } \gamma$$

This condition tells that the optimal isolated boundary is also the current line: the current density along it is constant.

For example, a rectangular domain is optimal if the potentials on two opposite sides of it are constants.

Example 7.3.3 (Optimal boundary component with the given potential) Suppose now that the boundary component γ_0 where the main boundary condition $w = C$ is imposed should be found from the optimality requirements. Again, we use the condition (7.25) combining it with the prescribed boundary conditions in the form

$$\frac{\partial u}{\partial t} = 0 \quad \text{on } \gamma_0, \text{ and } \gamma_1$$

The optimality of this component is expressed as

$$w_n = \sqrt{\lambda},$$

This condition implies the constancy of the normal current on the optimal boundary where the potential is constant.

Example 7.3.4 (Optimal boundary of the domain with given perimeter) Consider the previous conductivity problem in a domain with the boundary of the fixed perimeter but of arbitrary volume; this time

$$F = (\nabla w)^2, \quad f = \Lambda$$

where Λ is the Lagrange multiplier. The condition on the unknown boundary is

$$F + \Lambda k = 0 \quad \text{or } w_n^2 + w_t^2 + \Lambda k = 0$$

Combining it with the stationary condition

$$w_n = 0 \text{ on } \gamma$$

with respect to w, we obtain the condition

$$w_t^2 = \Lambda k.$$

which says that the square of the flux density along the boundary is proportional to its curvature.

Example 7.3.5 (Optimal shape of the membrane with minimal eigenfrequency) Consider the problem of the shape of a membrane of minimal eigenfrequency and given volume. Here

$$F = (\nabla w)^2 + c^2 w^2 + \Lambda, \quad f = 0$$

Here Λ is the Lagrange multiplier accounting for the volume constraint, w is deflection of the membrane which is zero on its boundary,

$$w = 0 \quad \text{on } \gamma.$$

The tangent derivative among the boundary $\frac{\partial w}{\partial t}$ is zero as well. The optimality condition becomes

$$\frac{\partial w}{\partial n} = \text{constant},$$

it shows that the normal derivative of the deflection is constant along the boundary. One easily guesses that the optimal shape is the circle.