Final HW 5500 Calculus of Variations

Spring 2015

1. Find Euler-Lagrange equation(s) and missing boundary conditions.

$$\min_{u(x)} \int_1^3 \left(x u'^2 - \frac{1}{x} u^2 \right) dx + u(3), \quad u(1) = 0 \tag{1}$$

2. Find Euler-Lagrange equation for the constrained problem. Solve.

$$\min_{u(x), v(x)} \int_0^1 (v\, u' - u)\, dx \tag{2}$$

where

$$\int_0^1 v^2(x)\, dx = 1, \quad u(0) = 0, \quad u(1) = 1$$

3. Find Euler-Lagrange equation and missing boundary conditions for the multivariable variational problem

$$\min_{u(x)} \int_\Omega \left((\nabla u)'^2 - c^2 u^2 \right) dx + a \int_{\partial\Omega} u^2 ds \tag{3}$$

where a and c are constants.

4. The Lagrangian L is

$$L(x, u, u') = a\, x u'^2 + 2b\, u\, u' + c\, u^2 \tag{4}$$

where a, b, c are constants.

– Find Hamiltonian, and dual Lagrangian.

– Write the Euler-Lagrange equations for L and the dual Lagrangian, and canonic system for the Hamiltonian.

5. Consider the problem

$$J_5 = \min_{u(x)} \int_0^1 L(x, u, u')dx, \quad u(0) = 0, \quad u(1) = 2 \tag{5}$$

where

$$L(x, u, u') = f(u') + (x - u)^2, \quad f(z) = \min\{z^2, z^2 + 2z - 1\} \tag{6}$$

– Check the Weierstrass condition.

– Derive the relaxed form of the variational problem, describe minimizing sequences.