Contents

Chapter 8

Multivariable problems: Scalar minimizer

8.1 Reminder of multivariable calculus

This part deals with multivariable variational problem that describe minimal surface areas, equilibria and dynamics of continua, optimization of shapes, etc. These problems require minimization of an integral of multivariable function and its gradient over a region Ω of a plane or space. The arguments of the Lagrangian are: Vector u of minimizers and the matrix $Du = \nabla u$ of gradients of these minimizers. Correspondingly, the analysis of these problem requires differentiation with respect to vectors and matrices. First, we remind of several formulas of vector (multivariable) calculus which will be commonly used in the next exposition.

8.1.1 Vector differentiation

Vector differentiation We remind the definition of the derivative.

Definition 8.1.1 If $\phi(\boldsymbol{a})$ is a scalar function of a column vector argument $\boldsymbol{a} = (a_1, \dots, a_n)^T$, then the derivative $\frac{d\phi}{d\boldsymbol{a}}$ is a row vector

$$\frac{d\phi}{d\boldsymbol{a}} = \left(\frac{d\phi}{da_1}, \dots, \frac{d\phi}{da_n}\right) \quad \text{if } a = \begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix} \tag{8.1}$$

assuming that all partial derivatives exist.

This definition comes from the consideration of for differential $d\phi$ of the scalar function $\phi(a)$:

$$\delta\phi(\boldsymbol{a}) = \phi(\boldsymbol{a} + d\boldsymbol{a}) - \phi(\boldsymbol{a}) = \frac{d\phi(\boldsymbol{a})}{d\boldsymbol{a}} \cdot d\boldsymbol{a} + o\|\boldsymbol{a}\|)$$

The left-had side is scalar and the second multiplier in the right-hand side is a column vector, therefore the first multiplier is a row vector defined in (8.1)

Examples of vector differentiation The next examples show the calculation of several commonly met functions. The results can be checked by straight calculations.

1. If $\phi(a) = \|a\|^2 = a_1^2 + \dots a_n^2$, the derivative is
$$\frac{d}{d\boldsymbol{a}}\boldsymbol{a}^2 = 2\boldsymbol{a}^T$$

2. The derivative of L_2 norm of $\boldsymbol{a}$ is
$$c = \frac{d}{d\boldsymbol{a}}\sqrt{\boldsymbol{a}^2} = 2\frac{\boldsymbol{a}^T}{\sqrt{\boldsymbol{a}^2}} = \frac{a^T}{|a|}$$
Observe that c is codirected with a and has unit length.

3. The derivative of the scalar product $c \cdot a$ where c is an n-dimensional vector is equal to c
$$\frac{d}{d\boldsymbol{a}}\boldsymbol{c}^T\boldsymbol{a} = \boldsymbol{c}$$
Similarly, the derivative of a product Ca where C is a $n \times k$ matrix equals C^T,
$$\frac{d}{da}Ca = C^T$$

4. The derivative of a quadratic form a^TCa where C is a symmetric matrix equals
$$\frac{d}{da}a^TCa = 2a^TC = 2(Ca)^T.$$

Gradient of a vector If $u = (u_1, \dots, u_n)$ is a vector function, $u_k = u_k(x_1, \dots x_d)$, then the gradient of u is defined as the $n \times d$ matrix denoted ∇u or Du, $\nabla u = \frac{\partial u_i}{\partial x_j}$, or, in elements,

$$\nabla u = Du == \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \dots & \frac{\partial u_n}{\partial x_1} \\ \dots & \dots & \dots & \dots \\ \frac{\partial u_1}{\partial x_d} & \frac{\partial u_2}{\partial x_d} & \dots & \frac{\partial u_n}{\partial x_d} \end{pmatrix} \tag{8.2}$$

The columns of this matrix are gradients of the elements of the vector u.

Directional derivative Let u_n be a directional derivative of a scalar function u in the direction n: $u_n = \nabla u \cdot n$. Partial derivative of $F(\nabla u)$with respect to u_n is defined as:
$$\frac{\partial F}{\partial u_n} = \frac{\partial F}{\partial \nabla u} \cdot n \tag{8.3}$$

8.1.2 Matrix differentiation

Similarly to the vector differentiation we define matrix differentiation considering a scalar function $\phi(A)$ of a matrix argument A. As in the vector case, the definition is based on the notion of the scalar product.

Definition 8.1.2 The scalar product a.k.a. the convolution of the $n \times m$ matrices A and $m \times n$ matrix B is defined as following

$$A : B = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ji}.$$

One can check the formula

$$A : B = \operatorname{Tr}(A\, B) \tag{8.4}$$

that brings the convolution into the family of the familiar matrix operations.

The convolution allows for calculation of increment of a *matrix-differentiable* function of a matrix argument caused by variation of this argument:

$$\delta\phi(A) = \phi(A + dA) - \phi(A) = \frac{d\phi(A)}{dA} : dA + o\|dA\|).$$

and to definition of the matrix-derivative:

Definition 8.1.3 The derivative of a scalar function ϕ by an $n \times m$ matrix argument A is an $m \times n$ matrix $D = \frac{d\phi}{dA}$ with elements

$$D_{ij} = \frac{\partial \phi}{\partial a_{ji}}$$

where a_{ij} is the ij-element of A.

In element form, the definition becomes

$$\frac{d\phi}{dA} = \begin{pmatrix} \frac{\partial\phi}{\partial a_{11}} & \frac{\partial\phi}{\partial a_{21}} & \dots & \frac{\partial\phi}{\partial a_{m1}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial\phi}{\partial a_{1n}} & \frac{\partial\phi}{\partial a_{2n}} & \dots & \frac{\partial\phi}{\partial a_{mn}} \end{pmatrix}$$

Examples of matrix differentiation Next examples show the derivatives of several often used functions of matrix argument.

1. As the first example, consider $\phi(A) = \operatorname{Tr} A = \sum_{i=1}^{n} a_{ii}$. Obviously,

$$\frac{d\phi}{da_{ij}} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

therefore the derivative of the trace is the unit matrix,

$$\frac{d}{dA} \operatorname{Tr} A = I.$$

2. Using definition of the derivative, we easily compute the derivative of a scalar product,
$$\frac{d(A:B)}{dA} = \text{ Tr } A^T B = B^T.$$
One can check that $A : B = \text{ Tr } A^T B$.

3. The derivative of the quadratic form $x^T A x = \sum_{i,j=1}^n x_i x_j a_{ij}$ is an $n \times n$ diad
$$\frac{d(x^T A x)}{dA} = x x^T$$

4. Compute the derivatives of the determinant of matrix A. Notice that the determinant linearly depends on each matrix element,
$$\det A = a_{ij} M_{ij} + \text{ constant}(a_{ij})$$
where M_{ij} is the minor obtained by elimination the ith row and the jth column of A; it is independent of a_{ij}. Therefore,
$$\frac{\partial \det A}{\partial a_{ij}} = M_{ij}$$
and the derivative of $\det A$ is the matrix M of minors of A,
$$\frac{d}{dA} \det A = M = \begin{pmatrix} M_{11} & \dots & M_{1n} \\ \dots & \dots & \dots \\ M_{1n} & \dots & M_{nn} \end{pmatrix}$$
Here, M_{ij} is the minor of the matrix A obtained by eliminating its ith row and jth column.

Recall that the inverse matrix A^{-1} can be conveniently expressed through these minors, $A^{-1} = \frac{1}{\det A} M$, and rewrite the result as
$$\frac{d}{dA} \det A = (\det A) A^{-1}$$
We may rewrite the result once more using the logarithmic derivative $\frac{d}{dx} \log f(x) = \frac{f'(x)}{f(x)}$. The derivative becomes more symmetric,
$$\frac{d}{dA} (\log \det A) = A^{-1}.$$

Remark 8.1.1 If A is symmetric and positively defined, we can bring the result to a perfectly symmetric form
$$\frac{d}{d \log A} (\log \det A) = I$$

if we introduce the matrix logarithmic derivative similarly to the logarithmic derivative of a real positive argument,

$$\frac{df}{d\log x} = x\frac{df}{dx},$$

which reads

$$\frac{df}{d\log A} = A\frac{df}{dA}.$$

Here, $\log A$ is the matrix that has the same eigenvectors as A and the eigenvalues equal to logarithms of the corresponding eigenvalues of A. (These logarithms when the eigenvalues of A are real and positive which in turn means that matrix A is symmetric and positively defined, as required above). Notice that $\log\det A$ is the sum of logarithms of eigenvalues of A,

$$\log\det A = \text{ Tr } \log A$$

.

5. Using the chain rule, we compute the derivative of the trace of the inverse matrix:

$$\frac{d}{dA} \text{ Tr } A^{-1} = -A^{-2}.$$

6. Similarly, we compute the derivative of the quadratic form associated with the inverse matrix:

$$\frac{d}{dA}x^T A^{-1} x = -xA^{-2}x^T.$$

Remark 8.1.2 (About the notations) The historical Leibnitz notation $\phi = \frac{\partial f}{\partial z}$ for partial derivative is not the most convenient one and can even be ambiguous. Indeed, the often used in one-variable variational problems partial $\frac{\partial f}{\partial u'}$ becomes in multivariable problem the partial of the partials $\frac{\partial u}{\partial x}$. Since there is no conventional analog for the symbol $'$ in partial derivatives, we need a convenient way to express the fact that the argument z of differentiation can itself be a partial derivative like $z = \frac{\partial u_1}{\partial x_2}$. If we were substitute this expression for z into $\frac{\partial f}{\partial z}$, we would arrive at an a bit awkward expression

$$\phi = \frac{\partial f}{\partial \frac{\partial u_1}{\partial x_2}}$$

(still used in Gelfand & Fomin) which replaces the expression $\frac{\partial f}{\partial u'}$ used in one-variable variational problem.

There are several ways to fix the inconvenience. To keep analogy with the one-variable case, we use the vector of partials $\frac{\partial f}{\partial(\nabla u)}$ in the place of $\frac{\partial F}{\partial u'}$. If needed, we specify a component of this vector, as follows

$$\phi = \left[\frac{\partial f}{\partial(\nabla u_1)}\right]_2$$

Alternatively, we could rename the partial derivatives of u with a simple indexed array D_{ij} arriving at the formula of the type

$$\phi = \frac{\partial f}{\partial D_{12}}, \quad \text{where } D_{12} = \frac{\partial u_1}{\partial x_2}.$$

or use comma to show the derivative

$$\phi = \frac{\partial f}{\partial u_{1,2}}, \quad \text{where } u_{1,2} = \frac{\partial u_1}{\partial x_2}.$$

The most radical and logical solution (which we do not dare to develop in the textbook) replaces Leibnitz notation with something more convenient, namely with Newton-like or Maple-like notation

$$\phi = D(f, D(u_1, x_2))$$

Remark 8.1.3 (Ambiguity in notations) A more serious issue is the possible ambiguity of partial derivative with respect to one of independent coordinates. The partial $\frac{\partial}{\partial x}$ means the derivative upon the explicitly given argument x of a function of the type $F(x, u)$. If the argument x is one of the independent coordinates, and if u is a function of these coordinates, in particular of x (as it is common in calculus of variations problems), the same partial could mean $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}$. To fix this, we need to specify whether we consider u as a function of x $u = u(x)$ or as an independent argument, which could make the notations awkward.

For this reason, we always assign the symbol x for a vector of independent variables (coordinates). When differentiation with respect to independent coordinates in considered, we use the gradient notations as ∇u. Namely, the vector is introduced

$$\nabla F(u, x) = \begin{pmatrix} \frac{\partial F}{\partial x_1} \\ .. \\ \frac{\partial F}{\partial x_d} \end{pmatrix} + \begin{pmatrix} \frac{\partial F}{\partial u}\frac{\partial u}{\partial x_1} \\ .. \\ \frac{\partial F}{\partial u}\frac{\partial u}{\partial x_d} \end{pmatrix}$$

where $\frac{\partial}{\partial x_k}$ always means the derivative upon explicit variable x. The partials corresponds to components of this vector. If necessary, we specify the argument of the gradient, as follows ∇_ξ.

8.1.3 Multidimensional integration

Change of variables Consider the integral

$$I = \int_\Omega f(x)\, dx$$

and assume that $x = x(\xi)$, or in coordinates

$$x_i = x_i(\xi_1, \dots, \xi_d), \quad i = 1, \dots, d$$

In the new coordinates, the domain Ω is mapped into the domain Ω_ξ and the volume element dx becomes $\det J \, d\xi$ where $\det J$ is the Jacobian and J is the $d \times d$ matrix gradient

$$J = \nabla_\xi x = \{J_{ij}\}, \quad J_{ij} = \frac{\partial x_i}{\partial \xi_j}, \quad i, j = 1, \ldots, d.$$

The integral I becomes

$$I = \int_{\Omega_\xi} f(x(\xi))(\det \nabla_\xi x) \, dx \tag{8.5}$$

The change of variables in the multivariable integrals is analogous to the one-dimensional case .

Green's formula The Green's formula is a multivariable analog of the Leibnitz formula a.k. the fundamental theorem of calculus. It has the form

$$\int_\Omega \nabla \cdot \boldsymbol{a} \, dx = \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n} \, ds \tag{8.6}$$

Here, $\boldsymbol{n}$ is the outer normal to Ω.

Integration by parts We will use multivariable analogs of the integration by parts. Suppose that $b(x)$ is a scalar differentiable function in Ω and $\boldsymbol{a}(x)$ is a vector differentiable field in Ω. Then the following generalization of integration by parts holds

$$\int_\Omega (\boldsymbol{a} \cdot \nabla b) \, dx = -\int_\Omega (b \nabla \cdot \boldsymbol{a}) \, dx + \int_{\partial\Omega} (\boldsymbol{a} \cdot \boldsymbol{n}) b \, ds$$

The formula follows from the differential identity (differentiation of a product)

$$\boldsymbol{a} \cdot \nabla b + b \nabla \cdot \boldsymbol{a} = \nabla \cdot (b \boldsymbol{a})$$

and Green's formula

$$\int_\Omega (\nabla \cdot \boldsymbol{c}) dx = \int_{\partial\Omega} (\boldsymbol{c} \cdot \boldsymbol{n}) \, ds$$

A similar formula holds for two differentiable in Ω vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\int_\Omega (\boldsymbol{a} \cdot \nabla \times \boldsymbol{b}) \, dx = \int_\Omega (\nabla \times \boldsymbol{a} \cdot \boldsymbol{b}) \, dx - \int_{\partial\Omega} (\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{n}) \, ds$$

It immediately follows from the identity

$$\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla \times \boldsymbol{a} - \boldsymbol{a} \cdot \nabla \times \boldsymbol{b}$$

and the Green's formula.

8.2 Euler equations for multiple integrals

8.2.1 Euler equation

Consider the simplest problem of multivariable calculus of variation: Minimize an integral over a regular bounded domain Ω with a smooth boundary $\partial\Omega$ of a twice differentiable Lagrangian $F(x, u, \nabla u)$ which depends on the minimizer u and its gradient ∇u and holds prescribed values $u_0(s)$ where s is (are) the coordinate(s) along the boundary $\partial\Omega$,

$$\min_{u: u\partial\Omega = u_0} I(u), \quad I(u) = \int_\Omega F(x, u, \nabla u) dx \tag{8.7}$$

As in the one-variable version, the Euler equation expresses the stationarity of the functional I to the variation of u. To derive the Euler equation, we consider the variation δu of the minimizer u and the difference $\delta I = I(u + \delta u) - I(u)$. For any minimizer u, this difference must be nonnegative, $\delta I(u, \delta u) \geq 0 \ \forall \delta u$.

Increment When the variation δu and its gradient are both infinitesimal and F is twice differentiable, we are allowed to linearize the perturbed Lagrangian:

$$\begin{aligned} F(x, u + \delta u, \nabla(u + \delta u)) \ &= \ F(x, u, \nabla u) + \frac{\partial F(x, u, \nabla u)}{\partial u} \delta u \\ &+ \ \frac{\partial F(x, u, \nabla u)}{\partial \nabla u} \delta \nabla u + o(|\delta u|, |\nabla \delta u|) \end{aligned}$$

Here, the term $\frac{\partial F(x,u,\nabla u)}{\partial(\nabla u)}$ denotes the vector partial derivatives of F with respect of partial derivatives of u,

$$\frac{\partial F(x, u, \nabla u)}{\partial(\nabla u)} = \left[\frac{\partial F(x, u, \nabla u)}{\partial\left(\frac{\partial u}{\partial x_1}\right)}, \ldots, \frac{\partial F(x, u, \nabla u)}{\partial\left(\frac{\partial u}{\partial x_n}\right)} \right].$$

Substituting this linearized expression into the expression for δI, we obtain

$$\delta I = \int_\Omega \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial \nabla u} \cdot \delta \nabla u \right) dx + o(|\delta u|, |\nabla \delta u|).$$

Interchanging two linear operators of variation and differentiation, $\delta \nabla u = \nabla \delta u$ and performing integration by parts of the second term in the above integrand:

$$\int_\Omega \left(\frac{\partial F}{\partial \nabla u} \cdot \delta \nabla u \right) dx = - \int_\Omega \delta u \left(\nabla \cdot \frac{\partial F}{\partial \nabla u} \right) dx + \int_{\partial\Omega} \delta u \left(\frac{\partial F}{\partial \nabla u} \cdot \boldsymbol{n} \right) ds$$

we obtain

$$\delta I \ = \int_\Omega S(F, u) \delta u \, dx + \int_{\partial\Omega} S_\partial(F, u, n) \delta u \, ds + o(|\delta u|, |\nabla \delta u|). \tag{8.8}$$

where

$$S(F,u) = \frac{\partial F}{\partial u} - \nabla \cdot \left(\frac{\partial F}{\partial \nabla u} \right) \tag{8.9}$$

– the coefficient by δu – is called the *variational derivative in* Ω or the *sensitivity function* and

$$S_\partial(F,u,n) = \frac{\partial F}{\partial \nabla u} \cdot n = \frac{\partial F}{\partial \left(\frac{\partial u}{\partial n}\right)} \tag{8.10}$$

is called the *variational derivative at* $\partial\Omega$. It is equal to the partial of F with respect to the normal derivative of u.

Stationarity The stationarity condition $\delta I = 0$ and the arbitrariness of variation δu in the domain Ω and possibly on its boundary $\partial\Omega$ lead to differential equation:

$$S(u,F) = 0 \quad \text{or } \nabla \cdot \frac{\partial F}{\partial \nabla u} - \frac{\partial F}{\partial u} = 0 \quad \text{in } \Omega \tag{8.11}$$

and the boundary condition

$$S_\partial(F,u,n)\, \delta u = 0 \quad \text{on } \partial\Omega \tag{8.12}$$

Notice that we leave δu is the expression for the boundary condition. This allows us to either assign u on the boundary or leave it free.

Main boundary conditions In the considered simplest problem, the partial differential equation (8.11) is integrated in Ω with the boundary conditions $u = u_0$. The boundary term (8.12) of the increment vanishes because the variation δu is zero (the value of u on the boundary is prescribed). This condition is called the *main boundary condition*. It is assigned independently of any variational requirements.

Natural boundary conditions If the value of u at the boundary is not fixed, the term (8.12) supplies the boundary conditions. When u on a boundary component is prescribed we say that the main boundary conditions are posed; in this case the variation of u on the boundary is zero, $\delta u = 0$ and (8.12) is satisfied. When no condition is prescribed on a boundary component, the natural condition

$$S_\partial(F,u,n) = \frac{\partial F}{\partial(\nabla u)} \cdot n = 0 \quad \text{or } \frac{\partial F}{\partial u_n} = 0 \tag{8.13}$$

(see (8.3)) must be hold. Notice, that the natural boundary condition appears from the minimization requirement; it must be satisfied to minimize the functional in (8.7).

Remark 8.2.1 The stationarity condition and the natural boundary conditions represent the stationarity (Euler-Lagrange equation) for multiple integrals. Notice that it is a direct analog of one-variable Euler equation: The derivative $\frac{d}{dx}$ with respect to the independent variable is replaced by ∇ or by $\nabla\cdot$. At the boundary, the derivative $\frac{du}{dx}$ is replaced by $\frac{\partial u}{\partial n} = \frac{\partial}{\partial \nabla u} \cdot n$. In the last case, the derivative with respect to x becomes the directional derivative along the normal to the boundary.

Remark 8.2.2 The existence of a solution to the boundary value problem (8.11), (8.12) in a bounded domain Ω requires ellipticity of the Euler equation. In turn, the ellipticity requires some properties of the Lagrangian which we will address later.

8.2.2 Examples of Euler-Lagrange equations

Here, we give several examples of Lagrangians, their Euler equations, and natural boundary conditions. We do not discuss the physics and do not derive the Lagrangians from some general principles of symmetry; this will be done later. Here, we simply introduce in algebraic derivation of the stationary equations.

Example 8.2.1 (Linear elliptic equation) The conductivity energy or a linear heterogeneous anisotropic material is

$$F = \frac{1}{2}(\nabla u)^T A(x)(\nabla u)$$

where $A(x) = \{A_{ij}(x)\}$ is a symmetric positively defined conductivity tensor which represent the material properties, and u is the potential like temperature or the concentration of conducting particles. The steady state distribution of the potential minimizes the total energy or to solve the variations problem (8.7) with the Lagrangian F. We comment on the derivation of this energy below in Section **??**. Here, we are concern with the form of the stationarity condition for this Lagrangian.

Compute the variational derivative. We have

$$\frac{\partial F}{\partial \nabla u} = A\nabla u, \quad S(u, F) = \nabla \cdot \frac{\partial F}{\partial \nabla u} = \nabla \cdot A\nabla u.$$

The stationarity condition (Euler equation) is the second-order elliptic equation:

$$S(u, F) = \nabla \cdot A(x)\nabla u = 0$$

or in the coordinates

$$S(u, F) = \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial u}{\partial x_j} = 0.$$

The natural boundary condition is $S_\partial(F, u, n) = n^T A\nabla u = 0$.

When A is proportional to the unit matrix I, $A = \kappa(x)I$, where $\kappa > 0$ is called the scalar conductivity, the Lagrangian becomes

$$F = \frac{\kappa(x)}{2}(\nabla u)^2$$

The Euler equation describes conductivity in a heterogeneous isotropic medium with a variable scalar conductivity constant $\kappa(x) > 0$. Its Euler equation is

$$\nabla \cdot \kappa(x) \nabla u = 0$$

or, in coordinate form

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} k(x) \frac{\partial u}{\partial x_i} = 0$$

and the natural boundary condition is $k(x)\frac{\partial u}{\partial n} = 0$ which is simplified to $\frac{\partial u}{\partial n} = 0$. It is called the *homogeneous Neumann condition*, while the principle boundary conditions $u = u_0$ on $\partial\Omega$ is called the *Dirichlet boundary condition*. Notice that a directional derivative, not the normal derivative, is zero on the boundary.

If the conductivity is constant, say $\kappa = 1$, Euler equation becomes the Laplace equation:

$$\nabla^2 u = 0;$$

it describes the stationary conductivity of a isotropic homogeneous medium.

Example 8.2.2 (Poisson and Helmholtz equations) Let us demonstrate that classical linear elliptic equations of mathematical physics originate from a variational problem of minimization of a quadratic Lagrangian. The Lagrangian in a form:

$$F = \frac{1}{2}(\nabla u)^2 - \frac{1}{2} a u^2 - bu \tag{8.14}$$

corresponds to the Euler equation

$$S(u, F) = \nabla^2 u + au + b = 0$$

which is called the inhomogeneous Helmholtz equation. The natural boundary condition $\frac{\partial u}{\partial n} = 0$ are independent on a and b.

If $a = 0$, the inhomogeneous Helmholtz equation degenerates into *Poisson equation*. If $b = 0$, it becomes homogeneous *Helmholtz equation*, and if $a = b = 0$ it degenerates into Laplace equation.

Example 8.2.3 (Nonlinear elliptic equation) Assume that the Lagrangian depends only on magnitude of the gradient:

$$F = \phi(|\nabla u|^2) \tag{8.15}$$

where ϕ is a monotonically increasing convex function, $\phi'(z) > 0$, $\forall z \in [0, \infty)$. Such Lagrangians describe the steady state conductivity or diffusion of an isotropic nonlinear material; u is the potential or concentration of diffusing particles.

The Euler equation is computed as

$$\nabla \cdot \left(\phi'(|\nabla u|^2) \nabla u\right) = 0$$

If $\phi' \geq 0$, the equation is elliptic. It is also can be rewritten as two first-order equations

$$\nabla \cdot j = 0, \quad j = \phi'(|\nabla u|^2)\, \nabla u = 0$$

where j is a divergencefree vector of *current*. The first equation express the equilibrium of the current density. The second equation is called the *constitutive relation*. It demonstrates the property of the material: the dependence of the current on the *field* ∇u. The coefficient $\phi'(|\nabla u|^2)$ is the conductivity of a nonlinear material; it depends on the magnitude of the field.

The natural boundary condition is $\phi' \frac{\partial u}{\partial n} = 0$ Because $\phi' > 0$, it simplifies to $\frac{\partial u}{\partial n} = 0$ and again expresses the vanishing of the normal derivative of u on the boundary.

In the next examples, we specify the function ϕ and obtain the variational form of well-studied nonlinear equations.

Example 8.2.4 (p-Laplacian) Consider the Lagrangian that corresponds to special nonlinearity $\phi(z) = pz^p$ in (8.16)

$$F = p|\nabla u|^p \tag{8.16}$$

The Euler equation is:

$$\nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right) = 0$$

The equation it is called p-Laplacian. It degenerates into Laplace equation when $p = 2$. Another interesting case $p = 1$. The Lagrangian becomes the norm of the gradient,

$$F = |\nabla u| = \sqrt{\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2} \tag{8.17}$$

(here, we consider for the definiteness the two-dimensional case). The corresponding Euler equation is:

$$\nabla \cdot \left(\frac{|\nabla u|}{\nabla u}\right) = a \quad \text{in } \Omega$$

It can be written as a system of two first-order partial differential equations

$$j = \frac{|\nabla u|}{\nabla u}, \quad \nabla \cdot j = a, \quad |j| = 1,$$

Observe that in this case $|j| = 1$ Here, the current j is codirected with ∇u and has the unit magnitude. As any unit vector, it admits the representation

$$j = (j_1, j_2), \quad j_1 = \cos\theta, \quad j_2 = \sin\theta$$

where $\theta(x)$ is an unknown scalar function, that is defined by the first-order equation $\nabla \cdot j = a$ or

$$-\sin\theta \frac{\partial \theta}{\partial x_1} + \cos\theta \frac{\partial \theta}{\partial x_2} = a$$

Potential u is found from another first-order equation that states that j is parallel ∇u, or $j \times \nabla u$=0. In the coordinate form, the equation becomes

$$\frac{\partial u}{\partial x_1} j_2 - \frac{\partial u}{\partial x_2} j_1 = 0 \quad \text{or} \quad \frac{\partial u}{\partial x_1} + (\cot \theta) \frac{\partial u}{\partial x_2} = 0$$

Notice the Euler equation is split into two first order equations.

A linear Null-Lagrangian Null-Lagrangians are important: the define the nonuniqueness of the problem of determination of the Lagrangian from the Euler equations. We discuss the null-Lagrangians for multivariable problems below in Section **??**. Here find the multidimensional linear null-Lagrangian.

Consider the Lagrangian $L_0(u) = h^T \nabla u + (\nabla \cdot h)u$, where h is a arbitrary differentiable vector field, and u is the scalar minimizer. L_0 depends on u linearly. It is easy to show that L_0 is a null-Lagrangian: Its Euler equation

$$\nabla \cdot \frac{\partial L_0}{\partial \nabla u} - \frac{\partial L_0}{\partial u} = \nabla \cdot h - \nabla \cdot h \equiv 0$$

is identically satisfied for all u. This Null-Lagrangian is similar to the linear null-Lagrangians in one-variable variational problems, see Section **??**. This is also evident from the identity (**??**) that allows for rewriting L_0 as $L_0 = \nabla \cdot (hu)$. By the divergence theorem, the minimizing functional is in fact the contour integral

$$L_0 dx = \nabla \cdot (hu) dx = \int_{\partial\Omega} u(h^T n) ds.$$

Therefore the Euler equation in the domain vanishes.

8.2.3 Smooth Approximation and continuation

As a first application of the theory of multivariable extremal problems, consider the problem of the approximation of data by a function u with assumed smoothness. The problem of approximation of a bounded, integrable, but may be discontinuous function $f(x)$ in R_3, by a smooth function $u(x)$ yields to the variational problem

$$\min_u \int_{R_n} \left((u - f)^2 + \epsilon^2 (\nabla u)^2 \right) dx$$

where term $\epsilon^2 (\nabla u)^2$ represents penalization. If $\epsilon \ll 1$, the first term of the integrand prevail, and u exactly approximates f. The ϵ grows, the approximation becomes less accurate but the u become more smooth. When $\epsilon \gg 1$, the approximation u tend to a constant function equal to the mean value of f.

The Euler equation for the approximation u is the inhomogeneous Helmholtz equation:

$$\epsilon^2 \nabla^2 u - u = f, \qquad \lim_{|x| \to \infty} u(x) = 0$$

This inhomogeneous Helmholtz problem can be explicitly solved through the resolvent representation:

$$u(x) = \int_{R^3} f(y) K(x-y) dy$$

Where $K(x-y)$ is the Green's function: Solution to the problem

$$(\epsilon^2 \nabla^2 - 1) K(r) = \delta(r), \quad \lim_{|r| \to \infty} u(r) = 0$$

The Green's function for the Helmholtz problem for the whole R^3 can be easily found in handbooks: It is

$$K(r) = \frac{1}{4\epsilon^2 \pi |r|} \exp\left(-\frac{|r|}{\epsilon}\right), \quad |r| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Using this representation, we obtain expression for u

$$u(x) = \frac{1}{4\epsilon^2 \pi} \int_{R_3} \exp\left(-\frac{|x-y|}{\epsilon}\right) \frac{f(y)}{|x-y|} dy$$

One observes that the smoothness of u is controlled by ϵ. When $\epsilon \to 0$, the kernel $K(r)$ becomes the delta-function, and $u(x) \to f(x)$.

Remark 8.2.3 Similar explicit solutions can be derived for R_2 and for some bounded domains, such as rectangles or circles (spheres).

Remark 8.2.4 In contrast with one-dimensional problems, the Green's function is unbounded but integrable.

Dealing with bounded domains, it is better to use the eigenfunction expansion, as it was explained in Section **??**.

A close problem is the analytic continuation.

Example 8.2.5 (Analytic continuation) Let $\Omega \subset R_2$ be a domain in a plane with a differentiable boundary $\partial\Omega$. Let $\phi(s)$ be a differentiable function of the point s of $\partial\Omega$. Consider the problem of *analytic continuation*: Find a function $u(x)$ in Ω such that it coincides with ϕ at the boundary, $u(s) = \phi(s)$, $\forall s \in \partial\Omega$ and minimizes the "roughness" inside Ω. To quantitatively measure the roughness we require that u minimizes the integral over Ω of $(\nabla u)^2$. Thus, we formulate a variational problem:

$$\min (\nabla u)^2 dx \ \text{ in } \Omega, \quad u|_{\partial\Omega} = \phi$$

Compute the stationarity conditions. We have

$$\frac{\partial (\nabla u)^2}{\partial \nabla u} = 2\nabla u, \quad \nabla \cdot \frac{\partial (\nabla u)^2}{\partial \nabla u} = 2\nabla \cdot \nabla u = 2\nabla^2 u = 0$$

which demonstrates that the minimizer must be harmonic in Ω or be a real part of an analytic function. This explains the name "analytic continuation".

Remark 8.2.5 Notice that the one-dimensional case is trivial: Ω is an interval, the boundary consists of two points, the minimizer is a straight line between these points. In that sense, harmonic functions are two-dimensional generalization of linear functions.

8.2.4 Change of coordinates

In order to transform the variational conditions to polar, spherical and other coordinates, consider the transformation of the independent variables $x = w(\xi)$ in a multivariable variational problem. Assume that the Jacobian of the transform is not zero in all points of Ω. In the new variables, the domain Ω becomes Ω_ξ, the differential dx is transformed as

$$dx = \det(J) d\xi$$

where J is the Jacobi matrix with the elements $J_{ij} = \{\frac{\partial w_i}{\partial \xi_j}\}$. By the chain rule, gradient $\nabla_x u$ in x coordinates becomes

$$\nabla_x u = \nabla_\xi u \frac{\partial \xi}{\partial x} = J^{-1} \nabla_\xi u$$

where ∇_ξ is the gradient in ξ-coordinates.

The integral

$$R = \int_\Omega F(x, u, \nabla u) dx$$

becomes

$$R = \int_{\Omega_\xi} F_\xi(\xi, u, \nabla_\xi u) d\xi$$

where F_ξ is defined as follows

$$F_\xi(\xi, u, \nabla_\xi u) = F(w(\xi), u, J^{-1}(\xi) \nabla_\xi u) \det J(\xi) \tag{8.18}$$

The Euler equation in the w-coordinates becomes $S(u, F_\xi) = 0$, where

$$S(u, F_\xi) = \nabla_\xi \cdot \frac{\partial F_\xi}{\partial \nabla_\xi u} - \frac{\partial F_\xi}{\partial u} \tag{8.19}$$

and the derivatives are related as

$$\frac{\partial F_\xi}{\partial u} = (\det J) \frac{\partial F}{\partial u} \quad \text{and} \quad \frac{\partial F_\xi}{\partial \nabla_\xi u} = (\det J) J^{-1} \frac{\partial F}{\partial \nabla u}$$

Example 8.2.6 (Helmholtz equation in polar coordinates) Let F be

$$F = \nabla^2 u + \alpha u^2 = u_x^2 + u_y^2 + \alpha u^2$$

and let pass to the polar coordinates $x = r\cos\theta$, $y = r\sin(\theta)$ and compute the Euler equation for F. We compute

$$J = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \quad \det J = r$$

Then

$$F_\xi = F_\xi(r,\theta,u,\nabla_\xi u) = r\left[\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 + \alpha\, u^2\right]$$

and the Euler equation (??) becomes

$$\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} - \alpha r u = 0$$

Example 8.2.7 (Laplacian in spherical coordinates)

8.2.5 First integrals

The first integrals in multivariable problems have some similarities to the first integrals in one-variable case.

Independence of the gradient of minimizer If the Lagrangian is independent of ∇u, $F = F(x, u)$, the Euler equation becomes an algebraic relation

$$\frac{\partial F}{\partial u} = 0.$$

As in one-dimensional case, the minimizer u does not need to be differentiable, even continuous function of x.

Independence of the minimizer If the Lagrangian is independent of u, $F = F(x, \nabla u)$ then Euler equation becomes

$$\nabla \cdot \frac{\partial F}{\partial \nabla u} = 0.$$

Instead of the constancy of $\frac{\partial F}{\partial u'}$ in one-dimensional case, here we state only the divergencefree nature of $\frac{\partial F}{\partial \nabla u}$ which implies the existence of a vector potential.

$$\frac{\partial F}{\partial \nabla u} = \nabla \times \psi \tag{8.20}$$

In the one-variable case, $\nabla \times \psi$ is replaced by a constant and we obtain the first integral; in multivariable case, the left-hand side of the previous equation is defined by two degrees of freedom (because ψ is defined up to a additive term $\nabla\phi$); no additional first integrals arrive.

Example 8.2.8 The one-dimensional analog of the two-dimensional Lagrangian $L_2 = (\nabla u)^2$ is the Lagrangian $L_1 = \left(\frac{du}{dx}\right)^2$ or $L_1 = (u')^2$. The Euler equation for the one-dimensional problem has the fist integral

$$\frac{\partial L_1}{\partial u'} = \frac{du}{dx} = C_1$$

followed by a solution $u = C_1 x + C_2$.

In multivariable case, we compute $\frac{\partial F}{\partial \nabla u} = 2\nabla u = V$. Here, we denote the gradient by $V = (v_1, v_2)$. The stationarity condition $\nabla \cdot V = 0$ or

$$\frac{\partial}{\partial x_1} v_1 + \frac{\partial}{\partial x_2} v_2 = 0$$

are identically satisfied if

$$v_1 = \frac{\partial \psi}{\partial x_2} \quad \text{and} \quad v_2 = -\frac{\partial \psi}{\partial x_1}$$

that is if (8.20) holds. ψ is called the adjoint potential, see below, Section ??. Instead of being a linear function as in one-dimensional case, the the minimizer u is harmonic – a solution to the Laplace equation $\nabla^2 u = 0$.

8.3 Variation of Boundary terms

8.3.1 Boundary integrals and Maier-Bolza problem

A natural extension of the simplest variational problem is the Meier-Bolza problem that asks for minimization of the sum of the volume and the integrals

$$I(u) = \min_u \left(\int_\Omega F(x, u, \nabla u) dx + \int_{\partial\Omega} f(s, u) ds \right) \tag{8.21}$$

This problem arrives both in physics when the boundary energy is taken into account and in optimization theory where the functionals could be of any form.

The increment of the functional consists of the bulk and boundary parts:

$$\delta I = \int_\Omega S(x, u, \nabla u) \delta u \, dx + \int_{\partial\Omega} B(s, u, u_n) \delta u \, ds$$

The boundary integral does not contribute to the bulk part and the Euler equation remains $S = 0$ where S is defined in (??). The boundary term B consists of the variation $\frac{\partial F}{\partial \nabla u} \cdot \boldsymbol{n} \delta u$ that is supplied by the variation of the volume integral, and of the variation $\frac{\partial f}{\partial u} \delta u$ supplied by the variation of the boundary integral:

$$B(s, u, u_n) \delta u = \left(\frac{\partial F}{\partial \nabla u} n + \frac{\partial f}{\partial u} \right) \delta u$$

The stationary condition is

$$\frac{\partial F}{\partial \nabla u} n + \frac{\partial f}{\partial u} = 0 \quad \text{on } \partial\Omega \tag{8.22}$$

Notice, that this condition degenerates into natural boundary condition (??) when the boundary term is zero, $f = 0$.Next example slightly generalizes the result demonstrating the variational problem that generates various classical types of boundary value problems for Laplace equation.

Variational origin of the Dirichlet, Neumann, and Robin problems

$$I(u) = \min_u \left[\frac{1}{2} \int_\Omega \nabla u^2 dx + \int_{\partial\Omega} \left(\frac{1}{2} a(s) u^2 + b(s) u \right) ds \right]$$

The Euler equation in the domain is the Laplace equation:

$$\nabla^2 u = 0 \quad \text{in } \Omega$$

The boundary condition is

$$\frac{\partial u}{\partial n} + a(s) u + b(s) = 0 \quad \text{on } \partial\Omega$$

This is the so-called radiation condition (Robin conditions) that specifies the rate of radiation $\frac{\partial u}{\partial n}$ depending on the value of u. When $a = b = 0$, the condition becomes an isolation condition (this is the natural condition for the bulk part of the problem); when $a = 0$ we deal with inhomogeneous Neumann conditions. Notice that the Dirichlet conditions $u = \beta(s)$ can be obtained as natural conditions in the limiting case when $b(s) = \beta(s) a(s)$ and $a(s)$ are arbitrary large, $a(s) \to \infty$.

Equilibrium of a loaded elastic body We revisit the linear elasticity problem, considering the equilibrium of an elastic body loaded by boundary and bulk forces. Assume that traction forces $f(s)$ are applied on the boundary ∂_1 of a domain Ω filled with a linear elastic material. The displacement u at the complementary component ∂_1 is prescribed. In addition, the bulk forces $q(x)$ like the gravity are applied in Ω. Then the equilibrium corresponds to the minimum of the sum of the whole elastic energy and the work of the bulk and boundary forces,

$$\min_u \left(W(\epsilon(\nabla u)) dx + w\, q\, dx + \int_{\partial_2} f^T u\, ds \right), \quad u|_{\partial_1} = u_0$$

(compare with (9.2). This time, we deal with Maier-Bolza problem for the vector minimizer u.

The stationarity is derived similarly to (9.3) and (9.4) correspond to the equilibrium of the loaded elastic material

$$\nabla \cdot \sigma = q, \quad \sigma = \frac{\partial W}{\partial \nabla u} = \mu \boldsymbol{\epsilon} + \lambda \operatorname{Tr} \boldsymbol{\epsilon} I \quad \text{in } \Omega, \tag{8.23}$$

These equations differ from (9.3) by an added bulk force q in the equilibrium condition. The boundary terms are

$$(\delta u)^T (\sigma n + f) = 0 \quad \text{on } \partial\Omega$$

Using the main boundary condition $u|_{\partial_1} = u_0$, we determine the mixed boundary conditions on $\partial\Omega$.

$$u = u_0 \text{ on } \partial_1, \quad \sigma n + f = 0 \text{ on } \partial_2. \tag{8.24}$$

Problem 8.3.1 Determine the boundary term in the Maier-Bolza problem which corresponds to the force f linearly depending on the displacement u on the boundary.

8.3.2 Examples

Inversion: Determination of the Lagrangian from Euler equation After the link between the variational problem and the boundary value problem is established, we can invert the situation and ask what variational functional corresponds to the given boundary value problem which we treat as the stationary condition to the unknown variational problem. Of course, we do not expect to obtain a unique solution. For instance, the null-Lagrangians cannot be accounted for. However, in many cases the variational problem can be easily guessed, as it is demonstrated at the next example.

Example 8.3.1 (Radiation of the black body) Consider the following inversion of the variational problem. Find the variational principle for the absolute black body with the radiation law $\frac{\partial u}{\partial n} = \gamma u^4$ at the boundary. It is assumed that the temperature u inside the body is harmonic. Using previous Example **??**, we easily guess the bulk and boundary terms in the Lagrangian:

$$I(u) = \min_u \int_\Omega \frac{1}{2}(\nabla u)^2 dx + \frac{\gamma}{5}\int_{\partial\Omega} u^5 ds$$

Example 8.3.2 (Relaxed analytic continuation) Let us return to the problem of analytic continuation, Example 8.2.5. We relax the problem's condition: Instead of prescribing the boundary data $u|_{\partial\Omega} = \phi(s)$ at every boundary point, we penalize solution its a deviation of the prescribed boundary value $\phi(s)$. Assuming that the penalty is proportional to the square of L_2 norm of the difference, we formulate the problem of *relaxed analytic continuation*:

$$\min_u I, \quad I = (\nabla u)^2 dx + \beta \int_{\partial\Omega} (u-\phi)^2 ds$$

where $\beta > 0$ is the penalization parameter. The stationarity condition is

$$\nabla^2 u = 0 \text{ in } \Omega, \quad \beta\frac{\partial u}{\partial n} + u = \phi \text{ on } \partial\Omega \tag{8.25}$$

Minimizer u satisfies Laplace equation with the boundary conditions of the third type, the so-called Robin problem. Notice that the minimizer tends to the minimizer of the problem in Example 8.2.5 if $\beta \to \infty$.

The solution allows for the following physical visualization. Imagine that u is the temperature. The problem in Example 8.2.5 describes the temperature distribution in a body with the fixed boundary temperature. The relaxed problem describes the temperature distribution in a body with the radiation from /absorbtion at the boundary. The rate of radiation is proportional to the difference $(u - \phi)$ between the fixed boundary temperature and target function and to β.

Example 8.3.3 (Relaxed continuation in circular domain) For demonstration of the relaxed analytic continuation, consider a circular domain $0 \le r \le 1$ and expand the boundary data into Fourier series

$$\phi(\theta) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\theta)$$

where

$$a_0 = 2\pi \int_{-\pi}^{\pi} \phi(\theta) d\theta, \quad a_k = \pi \int_{-\pi}^{\pi} \phi(\theta) \cos(k\theta) d\theta$$

are the known Fourier coefficients. The general solution of the Laplace equation in the circle has the form

$$u(r, \theta) = c_0 + \sum_{k=1}^{\infty} c_k r^k \cos(k\theta),$$

the coefficients c_k are found from the boundary condition in (8.25) as

$$c_0 = a_0, \quad \frac{k}{\beta} c_k + c_k = a_k$$

The solution to the relaxed continuation problem becomes

$$u(r, \theta) = a_0 + \sum_{k=1}^{\infty} \frac{\beta}{\beta + k} a_k \, r^k \, \cos(k\theta).$$

One observes that the coefficients by high harmonics ($k \gg 1$) in $u(r, \theta)$ are dumped due to relaxation. In other words, the problem of relaxed continuation is the continuation problem (Example 8.2.5) of continuation with smoothed boundary data ϕ_{smooth} instead of Gf. This smoothed data are given by the Fourier series

$$\phi_{\text{smooth}}(\theta) = a_0 + \sum_{k=1}^{\infty} \frac{\beta}{\beta + k} a_k \, r^k \, \cos(k\theta).$$

The solution to the stationary problem may not exist We remind that the stationarity is the necessary condition of the extremum. If the minimizer to the variational problem is continuous and differentiable, then it satisfies the stationary conditions. Next example shows that the Mayer-Bolza problem may lead to contradictory requirements so that the stationary solution of the problem does not exist.

Example 8.3.4 (Controversial boundary conditions) Consider the problem

$$I(u) = \min_u \int_\Omega |\nabla u| dx + \int_{\partial\Omega} au \, ds$$

or $F = |\nabla u|, \quad f = au$. The Euler equation in Ω was derived in the previous section, example **??**. The boundary condition is

$$\frac{\nabla u \cdot \boldsymbol{n}}{|\nabla u|} + a = 0$$

The first term in the left-hand side of the last expression is equal to the cosine of the angle between the normal n and the direction $\frac{\nabla u}{|\nabla u|}$ of the gradient; therefore the boundary condition becomes

$$\cos(\widehat{n, \nabla u}) = -a$$

The last condition is not controversial if $|a| \leq 1$. If $|a| = 1$, the direction of gradient coincide with the normal n, if $a = 0$, $\nabla u \cdot n = 0$. If $|a| \geq 1$, solution of the boundary values problem does not exists. We conjecture that the true minimizer is discontinuous and does not corresponds to Euler equation.

8.3.3 Weierstrass-Erdman condition

The Weierstrass-Erdman condition is satisfied on a surface where the gradient of an extremal is discontinuous; this discontinuity might be caused by a discontinuity in coefficients in the Lagrangian. Assume that Lagrangian is defined by two different expressions in two connected Ω_1 and Ω_2 parts of Ω:

$$\begin{array}{ll} F(x, u, \nabla u) = F_1(x, u, \nabla u) & \text{if } x \in \Omega_1; \\ F(x, u, \nabla u) = F_2(x, u, \nabla u) & \text{if } x \in \Omega_2. \end{array}$$

and consider the problem with the objective

$$I = \int_{\Omega_1} F_1(x, u, \nabla u) dx + \int_{\Omega_2} F_2(x, u, \nabla u) dx \tag{8.26}$$

The Weierstrass-Erdman condition is the boundary condition on the shared part ∂_{12} of the boundary between Ω_1 and Ω_2. At that surface, two conditions on u and $\frac{\partial u}{\partial n}$ are needed to uniquely continue the solution from one part to another.

The first (main) condition is the continuity of the differentiable potential u everywhere in Ω, including ∂_{12}.

$$[u]_-^+ = 0$$

This condition implies the continuity of the tangential derivative $\frac{\partial u}{\partial t}$ where t a tangent to ∂_{12}.

$$\left[\frac{\partial u}{\partial t}\right]_-^+ = 0 \quad \text{on } \partial_{12}$$

Indeed, the tangential derivative is calculated independently in the domains Ω_1 and Ω_2 as a limit of difference of potentials. Since the potential u is the same on the neighboring points on both sides, so is the difference between them.

Remark 8.3.1 In a three-dimensional problem, there are two independent tangential partial derivatives in two orthogonal tangential directions.

The second (variational) condition is called the Weierstrass-Erdman condition. It comes from the first variation of the objective (8.26)

$$\delta I = \int_{\Omega_1} S(F_1, u)\delta u\, dx + \int_{\Omega_2} S(F_2, u)\delta u\, dx$$
$$+ \int_{\partial\Omega_1} \frac{\partial F_1}{\partial \nabla u} n_1\, \delta u\, ds + \int_{\partial\Omega_2} \frac{\partial F_2}{\partial \nabla u} n_2\, \delta u\, ds$$

The stationarity of the integrals over Ω_1 and Ω_2 gives the Euler equations in that domains, and the boundary integrals give the variational boundary conditions. The variational condition on the shared boundary ∂_{12} involves both boundary integrals because the variation of δu is the same in both integrals, and the normals are codirected $n_1 = -n_2$. (Different signs of the normal correspond to the agreement that the normal is outer to the domain of variation. The outer normal to Ω_1 is the inner normal to Ω_2.) On the shared boundary ∂_{12}, we state the stationarity condition

$$\delta u : \int_{\partial_{12}} \left(\frac{\partial F_1}{\partial \nabla u} - \frac{\partial F_2}{\partial \nabla u} \right) \cdot n\, \delta u\, ds = 0$$

which results in the Weierstrass-Erdman condition

$$\frac{\partial F}{\partial \nabla u} \cdot n \Big|_{-}^{+} = 0 \quad \text{on } \partial_{12} \tag{8.27}$$

This condition is the direct analog of the Weierstrass-Erdman condition on the broken extremal in one-variable case (see Section **??**) that has the form $[\frac{\partial F}{\partial u'}]_{-}^{+} = 0$. In multivariate case, the Weierstrass-Erdman condition depends on the normal to the surface (line) of the discontinuity. The normal derivative of the minimizer can be discontinuous discontinuous. Simultaneously, its tangent derivative remains continuous to preserve the continuity of the surface.

Example 8.3.5 (Inhomogeneous conducting medium) The steady state conductivity of a medium corresponds to the variational problem with the Lagrangian $F = \frac{\kappa}{2}(\nabla u)^2$. where $\kappa = \kappa(x)$ is the conductivity. Assume that the medium is heterogeneous, and $\kappa(x)$ is a discontinuous function that takes two values κ_1 and κ_2 in Ω_1 and Ω_2, respectively,

$$\kappa(x) = \kappa_1 \chi(x) + \kappa_2(1 - \chi(x)) \tag{8.28}$$

where

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \Omega_1 \\ 0 & \text{if } x \notin \Omega_1 \end{cases} \tag{8.29}$$

Let us establish continuity conditions on the boundary $\partial_{12}\Omega$ between Ω_1 and Ω_2. We set $F_i = \frac{\kappa_i}{2}(\nabla u)^2$, $i = 1, 2$ and compute $\frac{\partial F}{\partial \nabla u} \cdot n = \kappa \frac{\partial u}{\partial n}$. The continuity conditions on the boundary $\partial_{12}\Omega$ become

$$\kappa \nabla u^T n|_{-}^{+} = 0, \quad \nabla u^T t|_{-}^{+} = 0,$$

or, in coordinates,

$$\frac{\partial u_-}{\partial t} = \frac{\partial u_+}{\partial t}, \quad \kappa_- \frac{\partial u_-}{\partial n} = \kappa_+ \frac{\partial u_+}{\partial n}$$

where u_- and u_+ are the minimizers in Ω_1 and Ω_2, respectively. These conditions allows for the following physical interpretation. The tangent component of the field u and the normal component of the current $j = \kappa \nabla u$ are continuous on the boundary between domains of different conductivity.

8.3.4 Effective conductivity of a laminate

The derived formulas allow for calculation of the *effective conductivity* of a laminate composite. Consider a periodic laminate submerged into a homogeneous field u_0. Let Ω be a unit square $\Omega = [0,1]\times[0,1]$ separated into $\Omega_1 = [0,1]\times[0,m]$ and $\Omega_2 = [0,1] \times [m,1]$. Call the components of the boundary by $\partial_1, \ldots, \partial_4$

$$\partial_1 = \{x: \quad x_1 = 0, \quad x_2 \in [0,1]\}, \quad \partial_2 = \{x: \quad x_1 = 1, \quad x_2 \in [0,1]\},$$
$$\partial_3 = \{x: \quad x_1 \in [0,1], \quad x_2 = 0\}, \quad \partial_4 = \{x: \quad x_1 \in [0,1], \quad x_2 = 1\}.$$

The conductivity corresponds to the minimization of the energy $W(V, \chi)$

$$W = \min_u \frac{1}{2} \kappa \nabla u^2 dx, \quad \kappa = \kappa_1 \chi(x) + \kappa_2 (1 - \chi(x)) \tag{8.30}$$

where χ is defined in (8.29), with the corresponding homogeneous boundary conditions $u|_{\partial\Omega} = V$.

In order to find the effective properties of the structure, we want to replace inhomogeneous distribution of the conductivity in Ω with a new equivalent material. Our goal is to express the energy as a quadratic form

$$W = \frac{1}{2} \nabla u_*^T \boldsymbol{\kappa}_* \nabla u_*, \quad \nabla u_* = \nabla u \, dx$$

of the average gradient ∇u_*, where u is the solution of (8.30). This way we determine the effective property κ_*. There are two cases that should be considered separately.

Case A. Let the field be applied across the layers. Assume that the main boundary conditions are

$$u = 0 \quad \text{if } x \in \partial_1, \quad u = V \quad \text{if } x \in \partial_2.$$

Here, V is the intensity of average field in the unit cell, $V = |\nabla u_*|$. The stationary condition in the domains Ω_i and the natural condition $\nabla u^T n = 0$ on the horizontal boundaries have the form

$$\kappa_i \nabla^2 u = 0 \text{ in } \Omega_i, \quad \frac{\partial u}{\partial x_1} = 0 \text{ on } \partial_3 \text{ and } \partial_4$$

They are satisfied if the potential is a continuous piece-wise linear function of x_1:

$$u(x_1) = A_1 x_1 x_2 \in [0,m] A_1 m + A_2 (x_1 - m) x_1 \in [m,1] \tag{8.31}$$

where A_1 and A_2 are related as follows

$$u(1) = A_1 m + A_2(1-m) = V,$$

The second condition for the constants A_1 and A_2, we use the Weierstrass-Erdman condition:

$$\kappa \frac{\partial u(m)}{\partial x_1}\bigg|_1^2 = \kappa_2 A_2 - \kappa_1 A_1 = 0$$

and find

$$A_1 = \frac{\kappa_2}{m\kappa_2 + (1-m)\kappa_1} V, \quad A_2 = \frac{\kappa_1}{m\kappa_2 + (1-m)\kappa_1} V$$

Substituting the found values of A_1, A_2 into (8.31) and compute the energy

$$W_A = m_1 \kappa_1 \nabla u^2|_{x\in\Omega_1} + (1-m_1)\kappa_2 \nabla u^2|_{x\in\Omega_1} = \frac{1}{2}\kappa_H V^2$$

where κ_H is the effective conductivity of the laminate across the layers,

$$\kappa_H = \left(\frac{m}{\kappa_1} + \frac{1-m}{\kappa_2}\right)^{-1}$$

Case B. Let the field be applied across the layers. The main boundary conditions are

$$u = 0 \quad \text{if } x_2 = 0, \quad x \in \partial_3, \quad u = V \quad x \in \partial_4$$

where V is the intensity of the average field, $|\nabla u_*| = V$. The stationary condition

$$\kappa_i \nabla^2 u = 0 \ \text{ in } \Omega_i, \quad \frac{\partial u}{\partial x_2} = 0 \text{ on } \partial_1 \text{ and } \partial_2$$

and the main boundary conditions are satisfied if potential u is a linear function of x_2:

$$u(x_2) = V x_2 \quad \text{in } \Omega_1 \text{ and } \Omega_2$$

The Weierstrass-Erdman condition is trivially satisfied because the field ∇u is parallel to the layers everywhere and $\frac{\partial u}{\partial n} = 0$. The energy W_B of this loading is

$$W_B = \frac{1}{2}\left(m_1 \kappa_1 \nabla u^2|_{x\in\Omega_1} + \kappa_2 \nabla u^2|_{x\in\Omega_1}\right) = \frac{1}{2}\kappa_A V^2$$

where

$$\kappa_A = m_1 \kappa_1 + (1-m_1)\kappa_2$$

is the effective conductivity of the laminate across the layers.

The results for the two loading cases are different from each other. This shows that the effective conductivity κ_* is anisotropic; it corresponds to a symmetric tensor

$$\kappa_* = \begin{pmatrix} \kappa_H & 0 \\ 0 & \kappa_A \end{pmatrix}$$

with eigenvalues κ_A and κ_H. The anisotropy is caused by the Weierstrass-Erdman conditions that introduce the dependence on the normal to the dividing boundary.

G-convergence If we consider the sequence of variational problems corresponding to a sequence of laminate layouts that are more and more fine but keep the volume fraction. The Lagrangians L^S of this sequence are

$$L^S = \frac{1}{2}\kappa^S(x)(\nabla u^S)^2$$

The limiting Lagrangian corresponds to the anisotropic effective material with the Lagrangian

$$L^0 = \frac{1}{2}\nabla u^0 \cdot \boldsymbol{\sigma}_* \nabla u^0$$

where $u^0 = \lim_{s\to\infty} u^S$ and $\boldsymbol{\sigma}_*$ is the tensor of effective conductivity with eigenvalues equal to κ_A and κ_H. One can see that the limiting Lagrangian changes its algebraic form. The convergence of the layouts $\kappa^s(x)$ to the tensorial layout $\boldsymbol{\sigma}_*$ is called G-convergence.

8.4 Isoperimetric problems

8.4.1 Constrained problems

The problems with isoperimetric constants are addressed similarly to the one-variable case. The extended functional is constructed in a usual way, by adding the constraint

$$\int_\Omega G(x, u, \nabla u)dx = 0$$

with a Lagrange multiplier λ to the Lagrangian

$$I = \int_\Omega F(x, u, \nabla u)dx, \quad u|_{\partial\Omega} = a(s).$$

The augmented Lagrangian is

$$\int_\Omega \left(F(x, u, \nabla u) + \lambda G(x, u, \nabla u)\right) dx, \quad u|_{\partial\Omega} = a(s). \tag{8.32}$$

Solution u of the Euler equation

$$\left(\nabla \cdot \frac{\partial}{\partial(\nabla u)} - \frac{\partial}{\partial u}\right)(F + \lambda G) = 0 \tag{8.33}$$

with natural boundary condition

$$n \cdot \frac{\partial}{\partial(\nabla u)}(F + \lambda G) = 0 \quad \text{on } \partial\Omega \tag{8.34}$$

depends on λ, $u = u(\lambda)$ which value is defined by the constraint.

Example 8.4.1 Consider the generalization of the problem of analytic continuation (Example 8.2.5): Find a function $u(x)$ that is equal to a given function ϕ on the boundary $\partial\Omega$ of Ω and minimizes the integral over the domain Ω of the Lagrangian $W = \frac{1}{2}(\nabla u)^2$. Generalizing the problem, we additionally require to keep the integral equality

$$\int_\Omega \rho\, u\, dx = 0 \tag{8.35}$$

where $\rho = \rho(x)$ is a weight function. The augmented functional is

$$I = \int_\Omega \left(\frac{1}{2}(\nabla u)^2 + \gamma\rho u\right) dx$$

where γ is the Lagrange multiplier by the constraint (8.35). The Euler equation is

$$\nabla^2 u - \gamma\rho = 0 \ \text{ in } \Omega, \quad u = \phi \ \text{ on } \partial\Omega$$

We solve this equation using the potential theory [?] treating γ as a known parameter and obtain $u = u(\gamma)$. Then, we compute integral of $\rho u(\gamma)$ and solve the equation (8.35) to γ.

The potential u has the representation

$$u(x) = \gamma E(x,\xi)\rho(\xi)d\xi + \int_{\partial\Omega} E(x,s)u_0(s)ds$$

where $E =$. Substituting this expression into (8.35), we find γ,

$$\gamma = -\frac{\rho(x)\left(\int_{\partial\Omega} E(x,s)u_0(s)ds\right)dx}{\rho(x)\left(E(x,\xi)\rho(\xi)d\xi\right)dx}$$

The problem is solved.

8.4.2 Incomplete boundary conditions

Integral constraints can be imposed on the boundary values of the minimizer if these values are determined from the optimality requirement, as the variational boundary conditions. The technique permits for addressing the problem in which the boundary conditions are not completely known, only some average information of them is available. This information must be accounted as the constraints to the boundary values of the minimizer, and the complete description comes from optimality requirement.

The Lagrange multipliers technique is used to formulate the problem for the extremal. The constraint

$$\int_{\partial\Omega} \phi(u,x)ds = 0$$

applied to the previous problem (8.33) (8.34) leads to the natural boundary conditions in the form

$$n \cdot \frac{\partial}{\partial(\nabla u)}(F + \lambda G) + \mu\frac{\partial\phi}{\partial u} \quad \text{on } \partial\Omega$$

The Euler equation is a complete boundary value problem that automatically incorporates the incomplete information about boundary condition and provides the bound for all admissible distribution of boundary values.

Example 8.4.2 (Incomplete conditions on the conducting rectangle) Consider the following problem. A rectangular domain Ω conducts electricity from the left $\partial\Omega_1$ to the right $\partial\Omega_2$ side, while the upper and lower sides $\partial\Omega_3$ and $\partial\Omega_4$ are insulated. The total current conducted through the domain is equal to j, but the details of the distribution of the boundary current is unknown.

We avoid the uncertainty requiring that the boundary currents would be "optimally" chosen to minimize the total energy of the domain. Assuming these boundary conditions, we formulate the variational problem

$$\min_u \int_\Omega (\nabla u)^2 \, dx$$

subject to constraints

$$\int_{\partial\Omega_1} u \, ds = j, \quad \int_{\partial\Omega_2} u \, ds = -j, \tag{8.36}$$

It is assumed here that the normal to Ω_1 coincides with the direction of the current, then the normal to the opposite side Ω_2 is opposite directed. We do not pose constraints to account for the insulation condition on the horizontal parts $\partial\Omega_3$ and $\partial\Omega_4$ because it is a natural condition that will be automatically satisfied at the optimal solution.

Accounting for the above constraints (8.36) with Lagrange multipliers λ_1 and λ_2 (that are related, as we show below), we end up with the variational problem

$$I = \min_u \int_\Omega \left[(\nabla u)^2 \, dx + \lambda_1 \left(\int_{\partial\Omega_1} u \, ds - j \right) + \lambda_2 \left(\int_{\partial\Omega_2} u \, ds + j \right) \right].$$

The calculation of the first variation of I with respect to u leads to the Euler equation in the domain

$$\nabla^2 u = 0 \quad \text{in } \Omega$$

and the boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial n} + \lambda_1 &= 0 \quad \text{on } \partial\Omega_1, \quad \frac{\partial u}{\partial n} + \lambda_2 = 0 \quad \text{on } \partial\Omega_2, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega_3, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_4 \end{aligned}$$

One can see that the boundary currents (the normal derivatives of the potential) are constant at the optimal solution. Finally, we exclude the Lagrange multipliers λ_1 and λ_2. The solvability condition

$$\int_{\partial\Omega_1} \frac{\partial u}{\partial n} \, ds + \int_{\partial\Omega_2} \frac{\partial u}{\partial n} \, ds = 0$$

(which is observed in the formulation of the constraints but is not exploited so far) yields to the equality $\lambda_1 + \lambda_2 = 0$.

The boundary current at $\partial\Omega_1$ and $\partial\Omega_2$ are constants that are calculated from the constraints,

$$\left.\frac{\partial u}{\partial n}\right|_{\Omega_1} = \frac{j}{l_1} \quad -\left.\frac{\partial u}{\partial n}\right|_{\Omega_2} = \frac{j}{l_2}$$

where l_1, l_2 are the lengths of the sides Ω_1 and Ω_2, $l_1 = \|\partial\Omega_1\|$ and $l_2 = \|\partial\Omega_2\|$.

The optimality requirement completely determines the problem requiring the constancy of the current density on the upper and lower sides. The magnitude of the current is adjusted to make the potential difference equal to two.

Problem 8.4.1 Consider the previous problem assuming that the domain Ω is a sector of the annulus $r_0 \le r_1$, $\theta_0 \le \theta \le \theta_1$ that conducts the fixed total current from one radial cut to another.

Problem 8.4.2 Consider the previous problem assuming that the conductivity is inhomogeneous. Derive the boundary conditions.

8.5 Lagrangian dependent on second derivatives

8.5.1 Stationarity conditions

Consider the problem

$$I(u) = \min_{u: u_{\partial\Omega} = u_0} \int_\Omega F(x, u, \nabla u, \nabla\nabla u) dx$$

with the Lagrangian that depends on the minimizer, its gradient, and its the Hessian matrix

$$H(u) = \nabla\nabla u = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & .. & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ .. & .. & .. \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & .. & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix}$$

The Euler equation for this problem are derived in the same way as before. First, compute the linear with respect to δu terms of the increment:

$$\delta I = \int_\Omega \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial \nabla u} \cdot \delta \nabla u + \frac{\partial F}{\partial \nabla\nabla u} : \delta \nabla\nabla u \right) dx$$

where the formula (**??**) for matrix differentiation is used.

The last two terms in the integrant are transformed by integrating them by parts so that they depend on δu but not of its graduent. The term $\frac{\partial F}{\partial \nabla u} \cdot \delta \nabla u$ is transformed as before by the means of the Green's formula (**??**)

$$\int_\Omega \left(\frac{\partial F}{\partial \nabla u} \cdot \delta \nabla u \right) dx = \int_\Omega \nabla \cdot \left[\frac{\partial F}{\partial \nabla} u \right] + \int_{\partial\Omega} \left(\frac{\partial F}{\partial \nabla u} \cdot n \right) \cdot \delta u \, ds$$

To transform the last term we apply the Green's formula twice:

$$\int_\Omega \left(\frac{\partial F}{\partial \nabla\nabla u} : \delta \nabla\nabla u \right) dx = \int_\Omega \left(\nabla\nabla : \frac{\partial F}{\partial \nabla\nabla u} \right) \delta u \, dx$$
$$+ \int_{\partial\Omega} \left[\left(\frac{\partial F}{\partial \nabla\nabla u} \cdot n \right) \cdot \delta \left(\frac{\partial u}{\partial n} \right) - \nabla \cdot \left(\frac{\partial F}{\partial \nabla\nabla u} \cdot n \right) \delta u \right] ds,$$

as a result, we obtain two boundary integrals proportional to $\delta u(s)$ and to $\delta \frac{\partial u}{\partial n}(s)$, respectively. Notice that these variations are mutually independent. Indeed, one can visualize $u(x)$ as a surface in three-dimensional space. It can be kept fixed at the boundary $u(s) = u_0$ but be approached from different angles, varying and vary the normal derivative $\frac{\partial u}{\partial n}(s)$.

Collecting the terms by independent variations $\delta u(x)$ in Ω and $\delta u(s)$ and $\delta \frac{\partial u}{\partial n}(s)$ on $\partial\Omega$, we obtain the increment as the sum of the integral I_Ω over Ω and the integral $I_{\partial\Omega}$ over $\partial\Omega$:

$$\delta I = \int_\Omega S(u, F) \, \delta u dx + \int_{\partial\Omega} \left(B_\alpha \, \delta u + B_\beta \, \delta \frac{\partial u}{\partial n}(s) \right) ds$$

where the bulk part $S(u, F)$ of the variational derivative is

$$S(u, F) = \frac{\partial F}{\partial u} - \nabla \cdot \left[\frac{\partial F}{\partial \nabla u} \right] + \nabla\nabla : \left[\frac{\partial F}{\partial \nabla\nabla u} \right]$$

and the boundary parts B_α and B_β of the derivative are

$$B_\alpha = \frac{\partial F}{\partial \nabla u} \cdot n - \nabla \cdot \left(\frac{\partial F}{\partial \nabla\nabla u} \cdot n \right) \quad \text{and } B_\beta = \frac{\partial F}{\partial \nabla\nabla u} : (n \, n^T)$$

The stationarity condition

$$S(u, F) = 0 \quad \text{in } \Omega \tag{8.37}$$

is the partial differential equation of the fourth order.

It is supplemented by the two boundary conditions

$$B_\alpha \, \delta u = 0 \quad \text{on } \partial\Omega \tag{8.38}$$
$$B_\beta \, \delta \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{8.39}$$

which required that one of the two multipliers in each condition is zero. The main conditions correspond to $\delta u = 0$ and $\delta \frac{\partial u}{\partial n} = 0$. The natural (variational) conditions correspond to $B_\alpha = 0$ and $B_\beta = 0$. There are also mixed cases.

To make notation clearer, we rewrite these equations in the coordinate form. Let us call h_{ij} the entries of the Hessian and g_k – the entries of the gradient of u. The vector of the normal n has the coordinates $n = (\cos \theta_1, \dots \cos \theta_n)$ where θ_i is tha angle between the normal and the coordinate axes. In these notations, we write $F = F(u, g_1, \ldots, g_d, h_{11}, \ldots, h_{dd})$. The variational derivative $S(u, F)$ becomes

$$S(u,F) = \frac{\partial F}{\partial u} - \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\frac{\partial F}{\partial g_k} + \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\frac{\partial^2 F}{\partial h_{ij}}$$

The boundary terms become

$$B_\alpha = \sum_{k=1}^{n} \cos\theta_k \frac{\partial F}{\partial g_k} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial^2 F}{\partial h_{ij}}\cos(\theta_j)\right)$$

and

$$B_\beta = \sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial h_{ij}} \cos(\theta_i)\,\cos(\theta_j)$$

The last two expression look awkward which is expected since the labor coordinate system is independent on the coordinates on the boundary. Alternatively, the boundary conditions can be rewritten in the local coordinates n, t on the boundary (we consider here the two-dimensional case for clarity in notations). The expressions for B_α and B_β are simplifies to

$$B_\alpha = \frac{\partial F}{\partial g_n} - \frac{\partial}{\partial n}\frac{\partial^2 F}{\partial h_{nn}} - \frac{\partial}{\partial t}\frac{\partial^2 F}{\partial h_{nt}} \quad \text{and } B_\beta = \frac{\partial^2 F}{\partial h_{nn}}$$

where $g_n = \frac{\partial u}{\partial n}$ and $h_{nn} = \frac{\partial^2 u}{\partial n^2}$.

The main conditions are always supplemented by the natural conditions so that the boundary value problem has two boundary conditions.

Remark 8.5.1 (Generalization) The generalization of the variational derivative to the lagrangian dependent of higher derivatives is straightforward. We leave this to the reader.

8.5.2 Bending plate

The energy Here we derive the energy W of an elastic *bending* surface. Consider the plane compact domain Ω with a differentiable boundary. Let $u(x)$ be the deflection of a point of the surface from the corresponding initial plane state, in the direction orthogonal to that plane. Assume that the energy W depends only on the tensor Υ of curvatures of the surface, which in a sense define the bending as the resistance to the shape change of a surface. The curvatures tensor for the surface $u(x_1, x_2)$ has the form (see [?])

$$\Upsilon = \frac{1}{(1 + (\nabla u)^2)^{\frac{3}{2}}} H \tag{8.40}$$

where H is the Hessian

$$H = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix}$$

it depends on the second derivatives of u.

Assume that the plate is isotropic and homogeneous: The energy is independent of the direction of the labor axes and on the position x. This property requires that the energy depends only on rotational invariants of Υ, or on its trace and determinant,

$$W = W(\operatorname{Tr}\Upsilon, \det\Upsilon)$$

Finally, assume that the surface is close to a plane so that the deflection u and all its derivatives are small,

$$u \ll 1, |\nabla u| \ll 1, \|\nabla\nabla u\| \ll 1$$

and the energy W is a differentiable function of its argument. Expand the energy W into Taylor series. In the expansion, the constant is of no interest because they it does not affect the stationarity. The linear terms are assumed to be zero

$$\left.\frac{\partial W}{\partial u}\right|_{u\equiv 0} = 0, \quad \left.\frac{\partial W}{\partial \nabla u}\right|_{u\equiv 0} = 0, \quad \left.\frac{\partial W}{\partial \nabla\nabla u}\right|_{u\equiv 0} = 0$$

because $u \equiv 0$ (no bending at all) must be a minimizer that corresponds to the global energy minimum.

There are only two quadratic terms: The square of the trace of Υ and its determinant. Notice that the quadratic terms of the curvatures Υ of u are approximated as the corresponding quadratic forms of Hessian H

$$\Upsilon = \frac{1}{(1 + (\nabla u)^2)^{\frac{3}{2}}} H = H + o((\nabla u)^2).$$

The energy W of the bending plate (up to the terms $o\left(|u|^2, |\nabla u|^2, \|\nabla\nabla u\|^2\|\right)$ becomes

$$W(H) = \frac{1}{2}\kappa(h_{x_1x_1} + h_{x_2x_2})^2 + \frac{1}{2}\rho(h_{x_1x_1}h_{x_2x_2} - h^2_{x_1x_2})$$

or

$$W(\nabla\nabla u) = \frac{1}{2}\kappa(\operatorname{Tr}[\nabla\nabla u])^2 + \frac{1}{2}\rho\det[\nabla\nabla u] \tag{8.41}$$

where κ and ρ are real constants and h_{ij} are the entries of the Hessian. We arrive at a surprising fact: the above mild and natural assumption determine the bending energy up to two real constants. Even more surprising (and may even be disappointing for the engineering students) is that the complicated equation of bending was derived without referring to the elasticity theory, but only to the geometrical features of surface bending and smallness of the deflection.

Remark 8.5.2 This energy is unsensitive to the shift of the deflection and to the small rotation of the plate in the three-dimensional space x_1, x_2, u,

$$F(u, \nabla u, \nabla\nabla u) = F(u + a, \nabla u + \boldsymbol{b}, \nabla\nabla u) \quad \forall a, \boldsymbol{b}.$$

Lagrangian The Lagrangian that determines the equilibrium of an elastic plate consist of its energy W, the work $q\,u$ of external loading by a distributed force $q = q(x)$ applied orthogonal to the plane of the plate, and the boundary terms. The bulk part of the Lagrangian is L is

$$L(u, \nabla\nabla u) = W(\nabla\nabla u) + q\,w$$

The boundary terms consist on the work $f\,u$ of the perpendicular to the plate forces $f = f(s)$ applied on the boundary, and the work of $M\,\frac{\partial u}{\partial n}$ of the bending momentums $M = M(s)$ applied on the boundary. The boundary part L_∂ of the Lagrangian is

$$L_\partial\left(u, \frac{\partial u}{\partial n}\right) = f\,u + M\,\frac{\partial u}{\partial n}$$

Equilibrium corresponds to minimum of the objective

$$L(u, \nabla\nabla u)\,dx + L_\partial\left(u, \frac{\partial u}{\partial n}\right)\,ds.$$

Euler equation To derive Euler equation we compute

$$\frac{\partial F}{\partial \nabla\nabla u} = \kappa\,(h_{11} + h_{22})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \rho\begin{pmatrix} h_{22} & -h_{12} \\ -h_{12} & h_{22} \end{pmatrix}, \quad \frac{\partial F}{\partial \nabla u} = 0.$$

The Euler equation is bi-Laplacian equation

$$S(u, F) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\kappa\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u + q = 0 \text{ in } \Omega \tag{8.42}$$

In nable-notations, it has the form:

$$\nabla^2\kappa\nabla^2 u = 0$$

Observe further that it does not depend on ρ. Indeed, the Euler equation $S_0(u)$ corresponding to the variation of the determinant,

$$\delta\int_\Omega (\rho \det H(u))dx = \int_\Omega S_0(u)\delta u\,dx, \tag{8.43}$$

where

$$S_0(u) = \left[\frac{\partial^2}{\partial x^2}\rho\left(\frac{\partial w^2}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\rho\left(\frac{\partial w^2}{\partial x^2}\right) - 2\frac{\partial^2}{\partial x \partial y}\rho\left(\frac{\partial^2 w}{\partial x \partial y^2}\right)\right]\delta u \equiv 0 \tag{8.44}$$

is identically zero because of equality of the mixed derivatives. In other words, $\det H(u)$ is a null-Lagrangian.

Remark 8.5.3 Notice that we found a null-Lagrangian $S_0(u)$ that is a nonlinear function of the derivatives of u. Such functionals do not exist in one-variable problem, as is was shown in Section ??. Here, the Euler equation vanishes because of integrability conditions, that is a pure multidimensional phenomenon. We investigate quadratic null-Lagrangians later in Section ?? where we develop the method to regularly find all of them.

First condition	Second condition	Comment
u is fixed	$\frac{\partial u}{\partial n}$ is fixed	Clamped
u is fixed	$M + B_\beta = 0$	Simply supported
$f + B_\alpha = 0$	$\frac{\partial u}{\partial n}$ is fixed	Sliding
$f + B_\alpha = 0$	$M + B_\beta = 0$	Free boundary

Boundary terms The boundary terms in the first variation are on $\partial\Omega$ are

$$\delta u(f + B_\alpha) + \delta\left(\frac{\partial u}{\partial n}\right)(M + B_\beta) = 0$$

where

$$B_\alpha = \frac{\partial}{\partial n}(B_\beta) + \frac{\partial}{\partial t}\left(\rho \frac{\partial^2 u}{\partial n \partial t}\right), \tag{8.45}$$

$$B_\beta = \kappa \frac{\partial^2 u}{\partial n^2} + (\kappa + \rho)\frac{\partial^2 u}{\partial t^2} \tag{8.46}$$

The natural boundary conditions do depend on ρ.

If the main conditions are given $\delta u = 0$ and $\delta\left(\frac{\partial u}{\partial n}\right) = 0$, the plate is called *clamped*. Its deflection on the boundary is fixed, and so is the normal angle. If one main condition $\delta u = 0$ is given, the plate is called simply supported. It is allowed to turn the plate around the point of support. The natural boundary condition $M + B_\beta = 0$ becomes active. It expresses the vanishing of the bending moment on the boundary of the plate.

Finally, let us comment on simplification of the expressions for B_α and B_β. Consider the simply supported boundary and assume that $u = 0$ along it. The tangent derivative $\frac{\partial u}{\partial t}$ is obviously zero as well. However, the second tangential derivative $\frac{\partial^2 u}{\partial t^2}$ is not zero if the boundary is curved. One can use Frenet formula

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial s^2} + k\frac{\partial}{\partial n}$$

where k is the curvature, to express the second tangential derivative through the second derivative $\frac{\partial^2 u}{\partial s^2} = 0$ along the arc of the boundary, which is zero because $u = 0$. The conditions on the simply supported boundary are

$$u = 0, \quad M + \kappa\frac{\partial^2 u}{\partial n^2} + (\kappa + \rho)k\frac{\partial u}{\partial n} = 0.$$

The other boundary conditions can be simplified in the same way.

Chapter 9

Multivariable problems: Vector minimizer

9.1 Several minimizers

9.1.1 Stationarity conditions

The next generalization is quite straightforward. Assume that Lagrangian depends on several potentials $\boldsymbol{u} = (u_1, \ldots, u_n)$ and on their derivatives: $n \times d$ Jacobian matrix

$$\nabla \boldsymbol{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \ldots & \frac{\partial u_n}{\partial x_1} \\ \ldots & \ldots & \ldots \\ \frac{\partial u_1}{\partial x_d} & \ldots & \frac{\partial u_n}{\partial x_d} \end{pmatrix}$$

as follows: $F = F(x, u, \nabla \boldsymbol{u})$.

The stationarity conditions are derived in the same way as in the case of one unknown function u. The variation of the functional is

$$\delta I = \left(\sum_{k=1}^{n} S(F, u_k) \delta u_k \right) dx + \int_{\partial\Omega} \left(\sum_{k=1}^{n} S_\partial(F, u_k, n) \delta u_k \right) ds$$

where $S(F, u_k)$ and $S_\partial(F, u_k, n)$ are the bulk and boundary parts of functional derivatives.

Stationarity corresponds to the system of Euler equations

$$S(F, u_k) = 0 \text{ in } \Omega, \quad S_\partial(F, u_k, n)\delta u_k = 0 \text{ on } \partial\Omega, \quad k = 1, \ldots, n$$

which express independency of the variation of each potential. The stationary conditions form a system of n second-order equations for n unknown potentials. Equations are of the same form as the Euler equations for the scalar case: scalar operations are simply replaced by vectorial ones and vectorial operations become matrix ones:

$$\nabla \cdot \frac{\partial F}{\partial (\nabla u)} - \frac{\partial F}{\partial u} = 0$$

where u is a vector $u = (u_1, \dots, u_n)$.

The coordinate form is

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k} \frac{\partial F}{\partial g_{ik}} - \frac{\partial F}{\partial u_i} = 0, \quad g_{ik} = \frac{\partial u_i}{\partial x_k}, \quad i = 1, \dots, n$$

Obviously this system degenerates into (8.11) when $n = 1$ and into (**??**) when $d = 1$.

The natural boundary conditions are

$$S_\partial(u_i, F) = \frac{\partial F}{\partial(\nabla u_i)} n = 0, \quad i = 1, \dots, n$$

or, in the coordinate form,

$$\frac{\partial F}{\partial g_{ik}} n_k - \frac{\partial F}{\partial u_i} = 0, \quad g_{ik} = \frac{\partial u_i}{\partial x_k}, \quad i = 1, \dots, n$$

where n_k is the projection of the normal to the axis x_k.

9.2 Elasticity

Elastic energy First, we argue on the form of elastic energy. Assume the following. The elastic equilibrium is defined by the vector u of displacements. The equilibrium corresponds to minimization of a function (the elastic energy) $W(\nabla u)$ of matrix ∇u. The stationarity relation are linear which implies that the energy is a homogeneous quadratic function of the elements of matrix ∇u.

The energy must be invariant to infinitesimal rotation of the labor system. Therefore, the energy is a quadratic function of the symmetric part of the displacement gradient

$$\epsilon(\nabla u) = \frac{1}{2} \left(\nabla u + (\nabla u)^T \right).$$

that is called *strain*. The supplementary antisymmetric part of gradient represents an infinitesimal rotation or the domain as a whole; such a motion which does not affect the elastic energy. The coordinate form of the strain is

$$\epsilon_{ii} = \frac{\partial u_i}{\partial x_i} \quad \text{and } \epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{if } i \neq j$$

The eigenvalues of strain are real.

The coefficients of the quadratic form $W(\nabla u)$ determine the material's elastic properties. Assume that the material is isotropic. It corresponds to the isotropic (rotationally invariant) quadratic form $W(\epsilon)$ of ϵ. Thus, the energy must be a function of the eigenvalues of ϵ. An the same time, it must be a quadratic function of the entries of this matrix. The general form of functions of a symmetric matrix ϵ with required properties is

$$W = \frac{1}{2} \mu \operatorname{Tr}(\boldsymbol{\epsilon}^2) + \frac{1}{2} \lambda (\operatorname{Tr} \boldsymbol{\epsilon})^2$$

where λ and μ are some real constants – elastic moduli of the material called also *Lamé moduli*. The coordinate form of the energy is

$$W = \frac{1}{2}\mu\left[\frac{1}{4}\sum_{i,j}^{d}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2\right]+\frac{1}{2}\lambda\left[\sum_{i}^{d}\frac{\partial u_i}{\partial x_i}\right]^2 \tag{9.1}$$

This derivation shows that the elastic properties of an arbitrary isotropic elastic material are defined by only two constants. Natural assumptions on the energy form based on expected linearity of the stationary condition and the symmetry principles lead to a unique form of the energy of a linear isotropic elastic material. The physical arguments are not directly used in this derivation.

Elasticity equations Let us find the stationarity conditions for the energy minimization. In physical terms, consider domain Ω filled with the linear isotropic elastic material. Assume that the displacements u are fixed on a component ∂_1 of the boundary and is free on the other boundary component ∂_2. The corresponding variational problem is

$$\min_u W(\epsilon(\nabla u))dx, \quad u|\partial_1 = u_0 \tag{9.2}$$

For convenience, we call $\sigma = \frac{\partial F}{\partial \nabla u}$ and notice that σ is a symmetric matrix because of the symmetry of the energy F called *stress* tensor. We write the Euler equation as the system

$$\nabla\cdot\sigma = 0, \quad \sigma = \frac{\partial W}{\partial \nabla u} = \mu\boldsymbol{\epsilon} + \lambda\,\mathrm{Tr}\,\boldsymbol{\epsilon} I \quad \text{in } \Omega, \tag{9.3}$$

and

$$(\delta\, u)^T \sigma\, n = 0 \quad \text{on } \partial\Omega \tag{9.4}$$

where I is the unit matrix. In elasticity σ is called the *stress tensor*, the first equation is called equilibrium condition, and the second – the Hook's law or the constitutive relations. Their coordinate forms are

$$\sum_i^d \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \; j = 1, \dots d, \quad \sigma_{ij} = \sigma_{ji},$$

– equilibrium conditions, and

$$\sigma_{ij} = \mu\,\epsilon_{ij}, \quad i \neq j, \quad i, j = 1, \dots d,$$
$$\sigma_{ii} = \mu\,\epsilon_{ii} + \lambda\sum_k \epsilon_{kk}, \quad i = 1, \dots d,$$

– Hooke's law.

The elasticity equations can be presented as the system of second-order equation called *Lamé equations*

$$\frac{\partial}{\partial x_i}\left[(\lambda+\mu)\frac{\partial u_i}{\partial x_i} + \mu\sum_{j\neq i}^{d}\frac{\partial u_i}{\partial x_j}\right] = 0, \quad i = 1, \dots, d$$

for the displacement vector u.

The main boundary conditions prescribe the displacement at the boundary $u(x) = u_0$ if $s \in \partial\Omega$. The natural boundary conditions

$$(\delta\, u)^T \sigma\, n = 0$$

require vanishing of a column (and, by the symmetry of σ, of a raw) of the stress tensor. Physically, they means the absence of the surface force at the boundary.

9.2.1 Other examples

Example 9.2.1 (Coupled conductivity) Consider the quadratic Lagrangian

$$F = \frac{1}{2}\nabla u_1 \cdot A_1 \nabla u_1 + \nabla u_1 \cdot A_{12} \nabla u_2 + \frac{1}{2}\nabla u_2 \cdot A_2 \nabla u_2 + \phi(u_1, u_2)$$

where A_1 and A_2 are positive symmetric matrices, and the whole system is positive:

$$\det \begin{pmatrix} A_1 & A_{12} \\ A_{12} & A_2 \end{pmatrix} > 0$$

The Euler equations are linear partial differential equations in the divergence form

$$\nabla \cdot (A_1 \nabla u_1 + A_{12} \nabla u_2) - \frac{\partial \phi}{\partial u_1} = 0$$
$$\nabla \cdot \left(A_{12}^T \nabla u_1 + A_2 \nabla u_2\right) - \frac{\partial \phi}{\partial u_2} = 0$$

They describe diffusion of two groups of particles that may transform to each other like groups of fast and slow neutrons in a nuclear reactor model. ϕ describes the recombination.

Example 9.2.2 (Polylinear null-Lagrangians) The vector problem admits a new type of nonlinear null-Lagrangians. A quadratic nonlinear (but polylinear) null-Lagrangian has the form

$$L_0 = \nabla u_1^T A_{12} \nabla u_2$$

where A_{12} is antisymmetric $a_{ij} = -a_{ji}$. We compute the Euler equation varying u_1

$$\nabla \cdot A_{12} \nabla u_2 = \sum_{i,j} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u_2}{\partial x_j} = \sum_{i,j} (a_{ij} + a_{ij}) \frac{\partial^2 u_2}{\partial x_i \partial x_j} \equiv 0 \quad \forall u_2$$

(the equation for u_1 is similar) Notice that the Euler equation is identically satisfied thanks to the integrability conditions (the equality of the mixed derivatives). This phenomenon is essentially multivariable, in is discussed below in Section ??.

Example 9.2.3 (A design problem) Find the Euler equations for the Lagrangian

$$L(u, \nabla u, R) = R(\nabla u)^2 + pR^p$$

where $u(x)$ and $R(x) > 0$ are the minimizers and $p > 1$ is a real parameter.

This variational problem arrives in a problem of optimal design of an inhomogeneous conducting material. We assume that u is the concentration of the diffusive material and $R(x)$ is the variable resistivity (the reciprocal to the conductivity κ) of the designed material. We further assume that the cost c of the material depends on its resistivity as follows $c = pR^p$. The design problem asks of the distribution of the resistivity $R(x)$ of the material in the domain that minimizes the total energy while keeping the materials cost low, or minimizes the sum of integrals of total energy and cost.

$$\min_{u,R} L(u, \nabla u, R)dx; \quad u = u_0 \text{ on } \partial\Omega.$$

Notice that the Lagrangian does not depends on ∇R. The stationarity with respect to R corresponds to the algebraic relation

$$\frac{\partial L}{\partial R} = -R^2(\nabla u)^2 + R^{p-1} = 0$$

or $R = |\nabla u|^{\frac{2}{p+1}}$. Substituting this value into L, we obtain

$$L(u, \nabla u, |\nabla u|^{\frac{2}{p+1}}) = |\nabla u|^{\frac{p+3}{2}} + p|\nabla u|^{\frac{2p}{p+1}}$$

This Lagrangian depends only on u and is of the type (8.16). Notice that the problem of the optimal distribution of the resistivity turns out to be equivalent to the resistivity of a nonlinear material. The nonlinearity occurs because the the optimal resistivity is proportional to the field ∇u.

9.2.2 Complement: Energy in nonlinear elasticity

Consider a body Ω and mark a point in Ω by a three-dimensional vector r. Assume that the body experience a *deformation* which brings the point r into a point $U(r)$. The mapping $U(r)$ is differentiable.

A particular mapping called *translation* is $U(r) = r + a$ where a is a constant vector. It shifts the coordinate system but does not change the shape of Ω. Another spacial mapping called *rotation* has the form $U = Rr$ where R is the rotation matrix, or $\det R = 1$, and $R^T = R^{-1}$. A rotation also does not change the shape of Ω. All other mapping do. Mechanics of continuum describes these other mappings, which the mechanics of a point describes translations and the mechanics of rigid bodies – the translations and the rotations.

Elasticity assumes that the deformation of a body minimizes a functional – the energy of deformation. The density F of the energy – the Lagrangian of the variational problem – determines the material properties. Let us discuss the form of the elastic energy.

Material independence The elastic energy is independent on the translation and rotation. These motions correspond to the positioning of the labor axes, not to material properties. The independency of the translation is achieved if we assume that the energy density depends only on the deformation gradient, $F =$

$F(\nabla U)$. When the coordinates are rotated, the gradient is redefined because the partials are computed in different directions. In a rotated coordinated, ∇U becomes $R\nabla U$. The energy density is independent of the rotation, if it depends on the *Cauchi deformation*

$$Ca(u) = [(\nabla u)^T(\nabla u)]^{\frac{1}{2}}$$

$F = F(Ca(u))$ Indeed, in the rotated coordinates, $(\nabla u)^T(\nabla u)$ remains invariant,

$$(\nabla u)^T(R^T R)(\nabla u) = (\nabla u)^T(\nabla u)$$

because $R^T R = I$.

Cauchi deformation is a symmetric positive definite matrix with the eigenvalues equal to the absolute values of the eigenvalues of ∇u.

Isotropy Dealing with isotropic materials, we assume in addition that the energy depends only on eigenvalues λ_i of $Ca(u)$ but not on its eigenvectors. Physically, this corresponds to the following requirement of isotropy: The same energy is stored in an isotropic material if a sphere of the material is cut, rotated, and clued to its place. because the eigenvalues are the roots of the characteristic equation for the matric $Ca(u)$, the energy is a function of the coefficients of this equation, or of the *main invariants*

$$\begin{aligned} i_1 = & \ \operatorname{Tr} Ca(U) = \lambda_1 + \lambda_2 + \lambda_3 \\ & i_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ & i_3 = \det Ca(U) = \lambda_1\lambda_2\lambda_3 \end{aligned}$$

of tensor $Ca(U)$:

$$F = F(i_1(Ca(U)), i_2(Ca(U)), i_3(Ca(U))$$

This is surprising: the energy of any nonlinear isotropic elastic material depends on only three scalar characteristics of the deformation.

Linear elasticity To derive equations for the linear elasticity, we assume that the *deflection* $u(r) = U(r) - r$ is small, or that the point after deformation remain close it their original position. The gradient of the deformation is close to the unit matrix, $\nabla U = \nabla u - I$, because $\|\nabla u\| \ll 1$. The Cauchi deformation becomes

$$\dashv(\nabla U) = [(\nabla U)^T(\nabla U)]^{\frac{1}{2}} = I + \frac{1}{2}(\nabla u + (\nabla u)^T = o\|\nabla u\|$$

The symmetric part of the deflection gradient is called the *strain* ϵ

$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T;$$

It defines the energy of any linear elastic body. The linearity of the equations require that the energy is expandable into Taylor series, and we keep only three first terms

$$F = F_0 + \psi : \epsilon + \frac{1}{2}\epsilon : C : \epsilon + o(\|\nabla u\|^2)$$

where ψ is a symmetric tensor called the *self-stress*, C is the fourth-order *stiffness* tensor: $C = \{c_{ijkl}\}$, and : means the contraction by two indices.

A quadratic form $x^T Ax$ for a vector is defined by a symmetric matrix A; similarly, the quadratic form for a symmetric matrix $\epsilon = \{\epsilon_{ij}\}$ is defined by a fourth-rank tensor C with special symmetries:

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}$$

Remark 9.2.1 The three-dimensional stiffness tensor C is determined by 21 constants. Indeed, an arbitrary array C_{ijkl} is defined by $3^4 = 81$ constant because each index can take of of three values. However, C defines a quadratic form and therefore is symmetric $C_{ijkl} = C_{jikl}$. The components ϵ_{ij} are also symmetric, $\epsilon_{ij} = \epsilon_{ji}$, with yields to the symmetries $C_{ijkl} = C_{ijlk}$ and $C_{ijkl} = C_{ijkl}$. These symmetries reduce the number of independent components of ϵ to six. The quadratic form over six-dimensional vector is defined by a 6×6 symmetric matrix is defined by $7 \cdot 6/2 = 21$ constants. Out of these constants, there are three angles of orientation of the labor system in space and eighteen are material characteristics that are invariant to the orientation of the labor system.

A similar consideration for the two-dimensional stiffness tensor shows that it depends on $4 \cdot 3/2 = 6$ constants. Five constants define the material properties, and one defined the angle of orientation of the labor system.

The Euler equations are

$$\nabla \cdot \sigma = 0, \quad \sigma = \sigma^T,$$
$$\sigma = C : \epsilon + \psi,$$
$$\epsilon = \frac{1}{2}\left((\nabla u) + (\nabla u)^T\right)$$

Here, $\sigma = \frac{\partial F}{\partial \nabla u}$ is called the *stress tensor*, its symmetry follows from the symmetry of $\epsilon(\nabla u)$, the linear relation between is called the constitutive relation, or, in elasticity, the generalized Hooke's law.

If the body Ω is not *prestressed* that is if the energy F is minimal when $u = 0$, the self-stress is zero, $\psi = 0$ and the generalized Hooke's law takes the form $\sigma = C : \epsilon$.

Isotropic linear elasticity Finally, we establish the form of the energy of an isotropic linear material. The requirements (**??**) and (**??**) together completely determine the first terms in the Taylor series expansion for energy of a homogeneous body. We may rewrite this expression in the form

$$F = \frac{1}{2}\mu \operatorname{Tr}(\boldsymbol{\epsilon}^2) + \frac{1}{2}\kappa(\operatorname{Tr}\boldsymbol{\epsilon})^2 + o(\|(\boldsymbol{\epsilon}\|^2)$$

which coincide with (9.2). This derivation shows that the listed above assumptions yield to the only variant of elasticity equations; the elastic properties of an arbitrary isotropic elastic material are defined by the two constants.

Existence of minimizers and stability Notice that the energy (9.1) is not the convex but polyconvex function of $\nabla \boldsymbol{u}$.

Explain the difference between elasticity equations and elliptic system. Why elasticity equations have two constants and elliptic system depends on three?

9.3 Stationarity of Lagrangians of Div and Curl

We start with the derivation of the Euler equations for Lagrangians of the type $L(u, \nabla \cdot u, \nabla \times u)$, where u is a vector minimizer. In the next section, we expand the technique to a general case of Lagrangians that depends on an arbitrary linear combination of partials of the vector potential.

Some formal identities Before deriving the equations, we explain the structure of the Curl and Divergence operators. Both of them are linear combinations of partials of a vector field. Consider a vector field $v = [v_1, v_2, v_3]^T$ in R^3 where v_i are differentiable function of x. The 3×3 matrix ∇v is a list of all partials

$$\nabla v = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}$$

The trace of this matrix is the divergence of v

$$\nabla \cdot v = \ \mathrm{Tr}\, \nabla v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_1} + \frac{\partial v_3}{\partial x_1}$$

The antisymmetric part $\nabla^A v = \frac{1}{2}(\nabla v - (\nabla v)^T)$ of ∇v is a matrix defined by three nonzero entrances

$$\nabla^A v = \frac{1}{2}(\nabla v - (\nabla v)^T) = \begin{pmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_3 & -c_1 & 0 \end{pmatrix}$$

defined by three nonzero entrances

$$c_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \quad c_2 = \frac{\partial v_1}{\partial x_1} - \frac{\partial v_3}{\partial x_1}, \quad c_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

These entrances form the vector $\nabla \times v = (c_1, c_2, c_3)$. The matrix is called the adjoint to $\nabla \times v$ (CHECK IT) matrix.

We also mention the identity

$$\nabla \times \nabla \times u = \nabla^2 u - \nabla \cdot \nabla u \tag{9.5}$$

for the second-order differential operations, that is easy to check using the integrability conditions.

Stationarity of a Lagrangian of Divergence Consider the Lagrangian $F(x, u, \nabla \cdot u)$ where $F(x, y, z)$ is the function of two d-dimensional vectors x and y and a scalar z. Consider the variational problem

$$\min_{\boldsymbol{u}} I, \quad I = \int_\Omega F(x, \boldsymbol{u}, \nabla \cdot \boldsymbol{u}) dx + \int_{\partial\Omega} f(s, \boldsymbol{u}) ds$$

where u is a minimizer, and f is a boundary Lagrangian.

The stationarity of I with respect to variation of a component u_i of the minimizer u gives the Euler equation

$$\frac{\partial}{\partial x_i}\frac{\partial F}{\partial z} - \frac{\partial F}{\partial u_i} \quad \text{in } \Omega, \quad \frac{\partial F}{\partial z} n_i + \frac{\partial f}{\partial u_i} \quad \text{on } \partial\Omega \tag{9.6}$$

where n_i is the ith component of the normal n, because F depends only of one partial $\frac{\partial u_i}{\partial x_i}$ of u_i. The stationarity with respect to the variation of the vector v corresponds to the vector equation

$$\nabla\left(\frac{\partial F}{\partial z}\right) - \frac{\partial F}{\partial u} = 0 \quad \text{in } \Omega, \quad n\frac{\partial F}{\partial z} + \frac{\partial f}{\partial u} = 0 \quad \text{on } \partial\Omega \tag{9.7}$$

where $z = \nabla \cdot u$ that has scalar equations (9.6) as components. The stationarity conditions form the system of the partial differential equations of the order $2d$ with d boundary conditions. The boundary conditions state that $\frac{\partial f}{\partial u}$ is parallel to the normal n. They can be rewritten as

$$\left|\frac{\partial f}{\partial u}\right| = \left(\frac{\partial F}{\partial z}\right)^2, \quad \left(\frac{\partial f}{\partial u}\right) \times n = 0$$

Problem 9.3.1 Derive Euler equations formally using the formula (??) for matrix differentiation and recalling that $\nabla \cdot u = \mathrm{Tr}\,(\nabla u)$.

Example 9.3.1 (Quadratic Lagrangian) Assume that Lagrangian has the form $L = \frac{1}{2}(\nabla \cdot u)^2 + \frac{1}{2}u^t u$ and $f = \frac{1}{2}u^2$ where u is a two-dimensional vector minimizer, and $\Omega \subset R^2$. The Euler equation is

$$\nabla(\nabla \cdot u) - u = 0, \quad n\nabla \cdot u = u$$

or, in coordinates,

$$\begin{aligned} \left(1 + \frac{\partial^2}{\partial x_1^2}\right) u_1 + \frac{\partial^2}{\partial x_1 \partial x_2} u_2 = 0, \quad n_1 \nabla \cdot u = u_1 \\ \left(1 + \frac{\partial^2}{\partial x_2^2}\right) u_2 + \frac{\partial^2}{\partial x_1 \partial x_2} u_1 = 0 \quad n_2 \nabla \cdot u = u_2 \end{aligned}$$

The boundary conditions are transformed to the form

$$|u|^2 = (\nabla \cdot u)^2, \quad n \times u = 0 \quad \text{on } \partial\Omega.$$

Dependence on Curl Consider the variational problem with Lagrangian $F(x, u, \nabla \times u)$, for a vector minimizer u,

$$\min_u I \quad \int_\Omega F(x, u, \nabla \times u) dx + \int_{\partial\Omega} f(s, u) ds.$$

We derive Euler equation is a standard manner. We compute the linearized increment of I as

$$I(u + \delta u) - I(u) = \int_\Omega \left(\frac{\partial F}{\partial u} + \frac{\partial F}{\partial(\nabla \times u)} \cdot \nabla \times (\delta u) \right) dx$$
$$+ \int_{\partial\Omega} \frac{\partial f(s, u)}{\partial u} ds + o(\|\delta u\|).$$

We integrate by parts the last term under bulk integral in the right-hand side using Stoks' theorem (**??**),

$$\int_\Omega \frac{\partial F}{\partial(\nabla \times u)} \cdot \nabla \times (\delta u) dx = -\int_\Omega \nabla \times \frac{\partial F}{\partial(\nabla \times u)} \cdot \delta u dx + \int_{\partial\Omega} \frac{\partial F}{\partial(\nabla \times u)} \times n(\delta u) ds$$

and arrive at the Euler-Lagrange equation:

$$\nabla \times \frac{\partial F}{\partial(\nabla \times u)} - \frac{\partial F}{\partial u} = 0 \quad \text{in } \Omega \tag{9.8}$$

and natural boundary condition

$$\frac{\partial F}{\partial(\nabla \times u)} \times n + \frac{\partial f}{\partial u} = 0.$$

Example 9.3.2 (Maxwell equations) Lagrangian for the Maxwell equation in vacuum is (see [])

$$\frac{1}{8\pi} \left\{ \left(\nabla A_0 - \frac{\partial \boldsymbol{A}}{\partial t} \right)^2 - \left(\nabla \cdot \boldsymbol{A} - \frac{\partial A_0}{\partial t} \right)^2 - (\nabla \times \boldsymbol{A})^2 \right\} \tag{9.9}$$

where $\boldsymbol{A}$ is the vector of magnetic potential and A_0 is the electric potential. We will comment on its derivation later, in Section **??**. Now we formally derive Euler equations.

Using the derived formulas, we obtain the stationarity conditions with respect to $\boldsymbol{A}$ and A_0 as

$$-\frac{\partial}{\partial t} \left(\nabla A_0 - \frac{\partial \boldsymbol{A}}{\partial t} \right) - \nabla \times \nabla \times \boldsymbol{A} - \nabla \left(\nabla \cdot \boldsymbol{A} - \frac{\partial A_0}{\partial t} \right) = 0$$

and

$$\nabla \cdot \left(\nabla A_0 - \frac{\partial \boldsymbol{A}}{\partial t} \right) + \frac{\partial}{\partial t} \left(\nabla \cdot \boldsymbol{A} - \frac{\partial A_0}{\partial t} \right) = 0$$

respectively.

After the simplification, and the use of (9.5) they take the canonic form of the Maxwell equations:

$$-\frac{\partial^2}{\partial t^2} \boldsymbol{A} + \nabla^2 \boldsymbol{A} = 0, \quad -\frac{\partial^2}{\partial t^2} A_0 + \nabla^2 A_0 = 0 \tag{9.10}$$

9.4 Pointwise constraints: Optimal design

The technique of dealing with pointwise algebraic or differential constraints is similar those used in to one-dimensional variational problems. The statement of the problem requires the definition of the goal functional, differential constraints (the equations of the equilibrium of dynamics), the possible integral constraints, and the set of controls.

Here we consider several simple minimization problems with differential constraints that express the thermal equilibrium. The equilibrium depends on the control (thermal sources or boundary conditions) that must be assigned to minimize the functional related to the temperature distribution. The relation between the temperature and the sources (the conductivity equation) is treated as differential constraint.

9.4.1 Design of boundary temperature

Consider the following problem: A bounded domain Ω is in thermal equilibrium. The temperature on its boundary $\theta(s)$ must be chosen to minimize the L_2 norm of deflection of the temperature T from a given target distribution $\rho(x)$ in a domain Ω.

Let us formally state the problem. The objective is

$$I = \min_T \frac{1}{2} \int_\Omega (T - \rho)^2 dx \tag{9.11}$$

where T is a solution to the boundary value problem of thermal equilibrium (the differential constraint)

$$\nabla^2 T = 0 \quad \text{in } \Omega, \quad T = \theta \quad \text{on } \partial\Omega. \tag{9.12}$$

This problem connects the control $\theta(s)$ with the state variable $T(x)$, while the objective depends on T. The set of controls is the open set of all ny piece-wise differentiable functions.

Harmonic target The problem become trivial when target ρ is harmonic, $\nabla^2 \rho = 0$ in Ω. In this case, we simply set $T = \rho$ everywhere in Ω and in particular at the boundary. The differential constraint is satisfied. The cost of the problem is zero, which mean that the global minimum is achieved.

Nonharmonic target We account for the first equation (9.12) as for the pointwise constraint. The Lagrange multiplier for the differential constraint (called also the adjoint variable) $\lambda(x)$ is a function of a point of the domain, because the constraint is enforced everywhere there. The augmented functional is

$$I_A = \int_\Omega \left(\frac{1}{2}(T - \rho)^2 + \lambda \nabla^2 T \right) dx. \tag{9.13}$$

The variation with respect to T in Ω gives

$$I_A = \int_\Omega \left(T - \rho + \nabla^2 \Lambda\right) \delta T \, dx + \oint_{\partial\Omega} \left[\left(\delta \frac{\partial T}{\partial n} \right) \lambda - \delta T \left(\frac{\partial}{\partial n} \lambda \right) \right] ds \qquad (9.14)$$

The variation leads to the stationarity condition – the boundary value problem for λ,

$$\nabla^2 \lambda + T = \rho \quad \text{in } \Omega, \qquad (9.15)$$

Notice that the variations (see (9.12)) of the value of T and its normal derivative $\frac{\partial T}{\partial n}$ at the boundary $\partial\Omega$ are arbitrary because the control θ is not constrained, therefore the coefficients by these variations must be zero at the stationary solution. We obtain

$$\lambda = 0, \quad \frac{\partial}{\partial n} \lambda = 0 \quad \text{on } \partial\Omega \qquad (9.16)$$

Observe that the problem for the dual variable λ has two boundary conditions, and the problem for T has none. These two problem are solved as a system of two second-order partial differential equations with two boundary conditions.

Remark 9.4.1 The primal problem is underdetermined because the control is not specified without the dual problem. The dual problem is overdetermined because it determines the control, and thus makes the pair of problem well-posed.

To solve the system of necessary conditions, we first exclude T by taking Laplacian ∇^2 of the left- and right-hand side of the equation (9.15) and accounting for (9.12). Thus, we obtain a regular fourth-order problem for λ

$$\nabla^4 \lambda = \nabla^2 \rho \quad \text{in } \Omega, \quad \lambda = 0, \quad \frac{\partial}{\partial n} \lambda = 0 \quad \text{on } \partial\Omega. \qquad (9.17)$$

that has a unique solution. After finding λ, we find T from (9.15). Then we compute the boundary values $\theta = T|_{\partial\Omega}$ and define the control. Notice that if the target is harmonic, $\nabla^2 \rho = 0$, the second term vanishes and $T = \rho$ as expected.

The computation goes as following. We define the Green's function $G(x, \xi)$ of the differential operator from the problem (9.17), or the solution to the boundary value problem

$$\nabla^4 G = \delta(\xi) \quad \text{in } \Omega, \quad G = 0, \quad \frac{\partial}{\partial n} G = 0 \quad \text{on } \partial\Omega. \qquad (9.18)$$

Then λ – the solution of (9.17) – is expressed as a convolution

$$\lambda = G * \nabla^2 \rho = \int_\Omega G(x, \xi) \nabla_\xi^2 \rho(\xi) d\xi \qquad (9.19)$$

where $*$ is sign of convolution and subindex ξ show the variable of differentiation. From (9.15) we find the integral representation of the optimal temperature distribution through the target function ρ,

$$T = \rho - \nabla^2 \left(G * \nabla^2 \rho\right) = \rho - \int_\Omega \nabla_x^2 G(x, \xi) \nabla_\xi^2 \rho(\xi) d\xi \qquad (9.20)$$

Remark 9.4.2 Symbolically, denoting the Green's function of biharmonic equation (9.18) by ∇^{-4}, we rewrite (9.20) as

$$T = \mathcal{L}\rho, \quad \mathcal{L} = I - \nabla^2\nabla^{-4}\nabla^2 \tag{9.21}$$

Observe that $\nabla^2 T \equiv 0$ so that the differential constraints are always satisfied.

9.4.2 Design of the bulk sources

Consider again problem (9.11) of the best approximation of the target temperature. This time, consider the control of bulk sources. Namely, assume that the heat sources $\mu = \mu(x)$ can be applied everywhere in the domain Ω but the boundary temperature is kept equal zero. Assume in addition, that the L_2 norm of the sources is bounded.

In this case, T is a solution to the boundary value problem (the differential constraint)

$$\nabla^2 T = \mu \quad \text{in } \Omega, \quad T = 0 \quad \text{on } \partial\Omega \tag{9.22}$$

and μ is bounded by an integral constraint

$$\frac{1}{2}\int_\Omega \mu^2 dx = A \tag{9.23}$$

but not pointwise. These constraints are accounted with Lagrange multipliers $\lambda(x)$ and γ, respectively. The extended functional depends on two functions T and μ,

$$\int_\Omega L(T,\mu)dx - \gamma A$$

where

$$L(T,\mu) = \frac{1}{2}(T-\theta)^2 + \lambda(\nabla^2 T - \mu) + \frac{1}{2}\gamma\mu^2 \tag{9.24}$$

The variations of L with respect to T and μ lead to stationary conditions. The stationarity with respect to T results in the boundary value problem for λ,

$$\nabla^2\lambda = T - \theta \quad \text{in } \Omega, \quad \lambda = 0 \quad \text{on } \partial\Omega \tag{9.25}$$

The stationarity with respect to variation of μ leads to the pointwise condition

$$\lambda = -\gamma\mu$$

that allows to exclude μ from (9.22). and obtain the linear system

$$\begin{aligned} \nabla^2 T &= -\frac{1}{\gamma}\lambda \quad \text{in } \Omega, \quad T = 0 \quad \text{on } \partial\Omega \\ \nabla^2\lambda &= T - \theta \quad \text{in } \Omega, \quad \lambda = 0 \quad \text{on } \partial\Omega \end{aligned}$$

and an integral constraint

$$\frac{1}{2}\int_\Omega \lambda^2 dx = \frac{A}{\gamma^2}.$$

This system could be solved for $T(x)$, $\lambda(x)$ and the constant γ, which would completely define the solution.

Problem 9.4.1 Reduce the system to one fourth-order equation as in the previous problem. Derive boundary conditions. Using Green's function, obtain the integral representation of the solution through the target.