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Chapter 4

Distinguishing minimum from maximum or saddle

Stationary conditions point to a possibly optimal trajectory but they do not answer the question of the sense of extremum. A stationary solution can correspond to minimum, local minimum, maximum, local maximum, of a saddle point of the functional. In this chapter, we establish methods aiming to distinguish local minimum from local maximum or saddle. In addition to being a solution to the Euler equation, the true minimizer satisfies necessary conditions in the form of inequalities. Here, we introduce the variational tests, Weierstrass and Jacobi conditions, that supplement each other examining various variations of the stationary trajectory.

4.1 Local variations

4.1.1 Legendre Tests

Consider again the simplest problem of the calculus of variations (**??**)

$$\min_{u(x), x\in[a,b]} I(u), \quad I(u) = \int_a^b F(x, u, u')dx, \quad u(a) = u_a, \ u(b) = u_b.$$

and assume that the function $u(x)$ satisfies the Euler equation and boundary conditions, hence the first variation δI is zero. Let us compute the increment of the objective caused by the variation (**??**), (**??**). This time, we expand F into Taylor series keeping the quadratic terms

$$\delta I = I(u + \delta u) - I(u) = \int_a^b (F(x, u + \delta u, u' + \delta u') - F(x, u, u'))dx$$
$$+ \int_a^b \left(\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} \right] \delta u + A\delta u^2 + 2B\delta u \, \delta u' + C(\delta u')^2 \right) dx, \quad (4.1)$$

where

$$A = \frac{\partial^2 F}{\partial u^2}, \quad B = \frac{\partial^2 F}{\partial u \partial u'}, \quad C = \frac{\partial^2 F}{\partial (u')^2}$$

and all derivatives are computed at the point x at the optimal trajectory $u(x)$.

The term in the brackets in the integrant in the right-hand side of (4.1) is zero because the Euler equation is satisfied. The variation δu is zero outside of the interval $[x, x+\varepsilon]$, and has magnitude $\rho(x) = \pm\varepsilon\left(x - (x_0 + \frac{1}{2}\varepsilon)\right)^2$ of the order of ε^2 in this interval. Its derivative $\delta u'$ is of the order of ε, therefore $|\delta u'| \gg |\delta u|$ as $\varepsilon \to 0$. We conclude that the last term in the integrant in the right-hand side of (4.1) dominates. The inequality $\iota > 0$ implies inequality

$$\frac{\partial^2 F}{\partial (u')^2} \geq 0 \tag{4.2}$$

which is called *Legendre condition* or *Legendre test.*

4.1.2 Weierstrass Tests

The Weierstrass test detects stability of a solution to a variational problem against a different kind of variations – the strong local perturbations. It also compares trajectories that coincide everywhere except a small interval where their derivatives significantly differ.

Suppose that u_0 is the minimizer of the variational problem (**??**) that satisfies the Euler equation (**??**). Additionally, u_0 should satisfy another test that uses a type of variation δu different from (**??**). The variation used in the Weierstrass test is an infinitesimal triangle supported on the interval $[x_0, x_0+\varepsilon]$ in a neighborhood of a point $x_0 \in (0,\ 1)$ (see **??**):

$$\Delta u(x) = \begin{cases} 0 & \text{if } \ x \notin [x_0, x_0+\varepsilon], \\ v_1(x-x_0) & \text{if } \ x \in [x_0, x_0+\alpha\varepsilon], \\ v_2(x-x_0) - \alpha\varepsilon(v_1 - v_2) & \text{if } \ x \in [x_0+\alpha\varepsilon, x_0+\varepsilon] \end{cases}$$

where the parameters α $(0 < \alpha < 1)$, v_1 and v_2 are related

$$\alpha v_1 + (1-\alpha) v_2 = 0. \tag{4.3}$$

to provide the continuity of $u_0 + \Delta u$ at the point $x_0 + \varepsilon$, or equality $\Delta u(x_0 + \varepsilon - 0) = 0$.

The considered variation (the Weierstrass variation) is localized and has an infinitesimal absolute value (if $\varepsilon \to 0$), but its derivative $(\Delta u)'$ is finite, unlike the Legendre variation:

$$(\Delta u)' = \begin{cases} 0 & \text{if } \ x \notin [x_0, x_0+\varepsilon], \\ v_1 & \text{if } \ x \in [x_0, x_0+\alpha\varepsilon], \\ v_2 & \text{if } \ x \in [x_0+\alpha\varepsilon, x_0+\varepsilon]. \end{cases} \tag{4.4}$$

Computing δI from (**??**) and rounding up to ε, we find that the inequality holds

$$\begin{aligned} \delta I = \varepsilon[\alpha F(x_0, u_0, u_0' + v_1) + (1-\alpha) F(x_0, u_0, u_0' + v_2) \\ -F(x_0, u_0, u_0')] + o(\varepsilon) \geq 0 \end{aligned} \tag{4.5}$$

for a minimizer u_0. Notice that we approximately replace $u_0 + \delta u_0$ with u_0 keeping only terms of the order of $O(1)$ in the varied integrand, but we have to count for different value of the derivative.

The last expression yields to the Weierstrass test and the necessary Weierstrass condition. Any minimizer $u(x)$ of (**??**) satisfies the inequality

$$\alpha F(x_0, u_0, u_0' + v_1) + (1-\alpha) F(x_0, u_0, u_0' + v_2) - F(x_0, u_0, u_0') \geq 0. \tag{4.6}$$

The reader may recognize in this inequality the definition of convexity, or the condition that the graph of the function $F(.,.,z)$ lie below the the chord in between the points there the chord meet the graph. The Weierstrass condition requires convexity of the Lagrangian $F(x,y,z)$ with respect to its third argument $z = u'$. The first two arguments $x,\ y = u$ here are the coordinates $x,\ u(x)$ of the testing minimizer $u(x)$. Recall that the tested minimizer $u(x)$ is a solution to the Euler equation.

Theorem 4.1.1 (Weierstrass test) A differentiable minimizer $u(x)$ of the simplest variational problem that solves Euler equation yields to convexity of the integrand $F(x,u,v)$ with respect of its third argument $v = u'$ when $x, u(x), u'(x)$ is an arbitrary point of the stationary trajectory.

The Weierstrass test is stronger than the Legendre test because the convexity implies the nonnegativity of the second derivative. It compares the optimal trajectory with larger set of admissible trajectories.

Example 4.1.1 Consider the Lagrangian

$$F(u,u') = [(u')^2 - u^2]^2$$

It is convex as a function of u' if $|u'| \geq |u|$. Consequently, the solution u of Euler equation

$$\frac{d}{dx}[(u')^3 - u^2 u'] + u(u')^2 - u^3 = 0, \quad u(0) = a_0,\ u(1) = a_1$$

or

$$(3(u')^2 - u^2)u'' - u((u')^2 + u^2) = 0 \quad u(0) = a_0,\ u(1) = a_1$$

corresponds to a local minimum of the functional if, in addition, the inequality $|u'(x)| \geq |u(x)|$ is satisfied in all points $x \in (0,1)$.

The Legendre test gives the inequality

$$u'^2 \geq \frac{2}{3} u^2$$

that is weaker than the Weierstrass condition.

Remark 4.1.1 Convexity of the Lagrangian does not guarantee the existence of a solution to variational problem. It states only that a differentiable minimizer (if it exists) is stable against fine-scale perturbations. However, the minimum may not exist at all or be unstable to other variations.

If the solution of a variational problem fails the Weierstrass test, then its cost can be decreased by adding infinitesimal centered wiggles to the solution. The wiggles are the Weierstrass trial functions, which decrease the cost. In this case, we call the variational problem ill-posed, and we say that the solution is unstable against fine-scale perturbations.

Example 4.1.2 Notice that Weierstrass condition is always satisfied in the geometric optics. The Lagrangian depends on the derivative as $L = \frac{\sqrt{1+y'^2}}{v(y)}$ and its second derivative

$$\frac{\partial^2 L}{\partial y'^{\,2}} = \frac{1}{v(y)(1+y'^2)^{\frac{3}{2}}}$$

is always nonnegative if $v > 0$. It is physically obvious that the fastest path is stable to short-term perturbations.

Example 4.1.3 Notice that Weierstrass condition is always satisfied in the Lagrangian mechanics. The Lagrangian depends on the derivatives of the generalized coordinates through the kinetic energy $T = \frac{1}{2}\dot{q}R(q)\dot{q}$ and its Hessian equals generalized inertia R which is always positive definite. Physically speaking, inertia does not allow for infinitesimal oscillations because they always increase the kinetic energy while potential energy is insensitive to them.

Weierstrass $\mathcal{E}$-function Weierstrass suggested a convenient test for convexity of Lagrangian, the so-called $\mathcal{E}$-function equal to the difference between the value of Lagrangian $L(x,u,\hat{z})$ in a trial point $u, z = z'$ and the tangent hyperplane $L(x,u,u') - (\hat{z} - u')^T \frac{\partial L(x,u,u')}{\partial u'}$ to the optimal trajectory at the point u, u':

$$\mathcal{E}(L(x,u,u',\hat{z}) = L(x,u,\hat{z}) - L(x,u,u') - (\hat{z} - u')^T \frac{\partial L(x,u,u')}{\partial u'} \tag{4.7}$$

Function $\mathcal{E}(L(x,u,u',\hat{z})$ vanishes together with the derivative $\frac{\partial \mathcal{E}(L)}{\partial \hat{z}}$ when $\hat{z} = u'$:

$$\mathcal{E}(L(x,u,u',\hat{z})|_{\hat{z}=u'} = 0, \quad \frac{\partial}{\partial \hat{z}}\mathcal{E}(L(x,u,u',\hat{z})|_{\hat{z}=u'} = 0.$$

According to the basic definition of convexity, the graph of a convex function is greater than or equal to a tangent hyperplane. Thereafter, the Weierstrass condition of minimum of the objective functional can be written as the condition of positivity of the Weierstrass $\mathcal{E}$-function for the Lagrangian,

$$\mathcal{E}(L(x,u,u',\hat{z}) \geq 0 \quad \forall \hat{z}, \ \forall x, u(x)$$

where $u(x)$ tested trajectory.

Example 4.1.4 Check the optimality of Lagrangian

$$L = u'^4 - \phi(u,x)u'^2 + \psi(u,x)$$

Figure 4.1: The construction of Weierstrass $\mathcal{E}$-function. The graph of a convex function and its tangent plane.

where ϕ and ψ are some functions of u and x using Weierstrass $\mathcal{E}$-function.

The Weierstrass $\mathcal{E}$-function for this Lagrangian is

$$\mathcal{E}(L(x,u,u',\hat{z}) = \left[\hat{z}^4 - \phi(u,x)\hat{z}^2 + \psi(u,x)\right]$$
$$-\left[u'^4 - \phi(u,x)u'^2 + \psi(u,x)\right] - (\hat{z}-u')(4u'^3 - 2\phi(u,x)u).$$

or

$$\mathcal{E}(L(x,u,u',\hat{z}) = (\hat{z}-u')^2\left(\hat{z}^2 + 2\hat{z}u' - \phi + 3u'^2\right).$$

As expected, $\mathcal{E}(L(x,u,u',\hat{z})$ is independent of an additive term ψ and contains a quadratic coefficient $(\hat{z}-u')^2$. It is positive for any trial function $\hat{z}$ if the quadratic

$$\pi(\hat{z}) = -\hat{z}^2 - 2\hat{z}u' + \phi - 3u'^2$$

does not have real roots, or if

$$\phi(u,x) - 2u^2 \le 0$$

If this condition is violated at a point of an optimal trajectory $u(x)$, the trajectory is nonoptimal.

Vector-Valued Minimizer The Legendre and Weierstrass conditions and can be naturally generalized to the problem with the vector-valued minimizer. If the Lagrangian is twice differentiable function of the vector $u' = z$, the Legendre condition becomes

$$He(F,z) \ge 0 \tag{4.8}$$

(see Section **??**) where $He(F,z)$ is the Hessian

$$He(F,z) = \begin{pmatrix} \frac{\partial^2 F}{\partial z_1^2} & \dots & \frac{\partial^2 F}{\partial z_1 \partial z_n} \\ \dots & \dots & \dots \\ \frac{\partial^2 F}{\partial z_1 \partial z_n} & \dots & \frac{\partial^2 F}{\partial z_n^2} \end{pmatrix}$$

and inequality in (4.8) means that the matrix is nonnegative definite (all its eigenvalues are nonnegative). The Weierstrass test requires convexity of $F(x,\boldsymbol{y},\boldsymbol{z})$ with respect to the last vector argument.

4.1.3 Null-Lagrangians and convexity

Find the Lagrangian cannot be uniquely reconstructed from its Euler equation. Similarly to antiderivative, it is defined up to some term called null-Lagrangian.

Definition 4.1.1 The Lagrangians $\phi(x, \boldsymbol{u}, \boldsymbol{u}')$ for which the operator $S(\phi, u)$ of the Euler equation (??) identically vanishes

$$S(\phi, u) = 0 \quad \forall u$$

are called *Null-Lagrangians*.

Null-Lagrangians in variational problems with one independent variable are linear functions of $\boldsymbol{u}'$. Indeed, the Euler equation is a second-order differential equation with respect to $\boldsymbol{u}$:

$$\frac{d}{dx}\left(\frac{\partial}{\partial \boldsymbol{u}'}\phi\right) - \frac{\partial}{\partial \boldsymbol{u}}\phi = \frac{\partial^2 \phi}{\partial (\boldsymbol{u}')^2} \cdot \boldsymbol{u}'' + \frac{\partial^2 \phi}{\partial \boldsymbol{u}' \partial \boldsymbol{u}} \cdot \boldsymbol{u}' + \frac{\partial^2 \phi}{\partial \boldsymbol{u} \partial x} - \frac{\partial \phi}{\partial \boldsymbol{u}} \equiv 0. \qquad (4.9)$$

The coefficient of $\boldsymbol{u}''$ is equal to $\frac{\partial^2 \phi}{\partial (\boldsymbol{u}')^2}$. If the Euler equation holds identically, this coefficient is zero, and therefore $\frac{\partial \phi}{\partial \boldsymbol{u}'}$ does not depend on $\boldsymbol{u}'$. Hence, ϕ linearly depends on $\boldsymbol{u}'$:

$$\begin{aligned} \phi(x, \boldsymbol{u}, \boldsymbol{u}') &= \boldsymbol{u}' \cdot A(\boldsymbol{u}, x) + B(\boldsymbol{u}, x); \\ A &= \tfrac{\partial^2 \phi}{\partial \boldsymbol{u}' \partial \boldsymbol{u}}, \quad B = \tfrac{\partial^2 \phi}{\partial \boldsymbol{u} \partial x} - \tfrac{\partial \phi}{\partial \boldsymbol{u}}. \end{aligned} \qquad (4.10)$$

If, inn addition, the following equality holds

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial \boldsymbol{u}}, \qquad (4.11)$$

then the Euler equation vanishes identically. In this case, ϕ is a null-Lagrangian.

We notice that the Null-Lagrangian (4.10) is simply a full differential of a function $\Phi(x, u)$:

$$\phi(x, \boldsymbol{u}, \boldsymbol{u}') = \frac{d}{dx}\Phi(x, u) = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u} u';$$

equations (4.11) are the integrability conditions (equality of mixed derivatives) for Φ. The vanishing of the Euler equation corresponds to the Fundamental theorem of calculus: The equality

$$\int_a^b \frac{d\Phi(x, u)}{dx} dx = \Phi(b, u(b)) - \Phi(a, u(a)).$$

that does not depend on $u(x)$ only on its end-points values.

Example 4.1.5 Function $\phi = u\, u'$ is the null-Lagrangian. Indeed,we check

$$\frac{d}{dx}\left(\frac{\partial}{\partial u'}\phi\right) - \frac{\partial}{\partial u}\phi = u' - u' \equiv 0.$$

Remark 4.1.2 We will show in Section ?? that nonlinear null-Lagrangians in multivariable problems exist that express the integrability conditions.

Null-Lagrangians and Convexity The convexity requirements of the Lagrangian F that follow from the Weierstrass test are in agreement with the concept of null-Lagrangians (see, for example [**?**]).

Consider a variational problem with the Lagrangian F,

$$\min_{\boldsymbol{u}} \int_0^1 F(x, \boldsymbol{u}, \boldsymbol{u}')dx.$$

Adding a null-Lagrangian ϕ to the given Lagrangian F does not affect the Euler equation of the problem. The family of problems

$$\min_{\boldsymbol{u}} \int_0^1 \left(F(x, \boldsymbol{u}, \boldsymbol{u}') + t\phi(x, \boldsymbol{u}, \boldsymbol{u}')\right) dx,$$

where t is an arbitrary number, corresponds to the same Euler equation. Therefore, each solution to the Euler equation corresponds to a family of Lagrangians $F(x, \boldsymbol{u}, \boldsymbol{z}) + t\phi(x, \boldsymbol{u}, \boldsymbol{z})$, where t is an arbitrary real number. This says, in particular, that a Lagrangian cannot be uniquely defined by the solution to the Euler equation.

The stability of the minimizer against the Weierstrass variations should be a property of the Lagrangian that is independent of the value of the parameter t. It should be a common property of the family of equivalent Lagrangians. On the other hand, if $F(x, \boldsymbol{u}, \boldsymbol{z})$ is convex with respect to $\boldsymbol{z}$, then $F(x, \boldsymbol{u}, \boldsymbol{z}) + t\phi(x, \boldsymbol{u}, \boldsymbol{z})$ is also convex. Indeed, $\phi(x, \boldsymbol{u}, \boldsymbol{z})$ is linear as a function of $\boldsymbol{z}$, and adding the term $t\phi(x, \boldsymbol{u}, \boldsymbol{z})$ does not affect the convexity of the sum. In other words, convexity is a characteristic property of the family. Accordingly, it serves as a test for the stability of an optimal solution.

4.2 Jacobi condition

4.2.1 Sufficient condition for the weak local minimum

We assume that a trajectory $u(x)$ satisfies the stationary conditions and Legendre condition. We investigate the increment caused by a nonlocal variation δu of an infinitesimal magnitude:

$$\mathcal{N}_{\text{Jacobi}}(\delta u) = \mathcal{N}_1(\delta u) + \mathcal{N}_2(\delta u) < \varepsilon, \quad \mathcal{N}_3(\delta u) \text{ is arbitrary.}$$

To compute the increment, we expand the Lagrangian into Taylor series keeping terms up to $O(\epsilon^2)$. Recall that the linear of ϵ terms are zero because the Euler equation $S(u, u') = 0$ for $u(x)$ holds. We have

$$\delta I = \int_0^r S(u, u')\delta u\, dx + \int_0^r \delta^2 F dx + o(\epsilon^2) \tag{4.12}$$

where

$$\delta^2 F = \frac{\partial^2 F}{\partial u^2}(\delta u)^2 + 2\frac{\partial^2 F}{\partial u \partial u'}(\delta u)(\delta u') + \frac{\partial^2 F}{\partial (u')^2}(\delta u')^2 \tag{4.13}$$

No variation of this kind can improve the stationary solution if the quadratic form

$$Q(u,u') = \begin{pmatrix} \frac{\partial^2 F}{\partial u^2} & \frac{\partial^2 F}{\partial u \partial u'} \\ \frac{\partial^2 F}{\partial u \partial u'} & \frac{\partial^2 F}{\partial (u')^2} \end{pmatrix}$$

is positively defined,

$$Q(u,u') > 0 \quad \text{on the stationary trajectory } u(x) \tag{4.14}$$

This condition is called the sufficient condition for the weak minimum because it neglects the relation between δu and $\delta u'$ and treats them as independent trial functions. If the sufficient condition is satisfied, no trajectory that is smooth and sufficiently close to the stationary trajectory can increase the objective functional of the problem compared with the objective at that tested stationary trajectory.

Notice that the term $\frac{\partial^2 F}{\partial u'^2}$ is nonnegative because of the Legendre condition (**??**).

Example 4.2.1 Show that the sufficient condition is satisfied for the Lagrangians

$$F = \frac{1}{2}u^2 + \frac{1}{2}(u')^2 \text{ and } F_2 = \frac{1}{|u|}(u')^2$$

Next example shows that violation of the sufficient conditions can yield to nonexistence of the solution.

Example 4.2.2 (Stationary solution is not a minimizer) Consider the variational problem:

$$I = \min_u \int_0^r \left(\frac{1}{2}(u')^2 - \frac{c}{2}u^2\right) dx \quad u(0) = 0; \quad u(r) = A$$

where c is a constant. The first variation δI is zero,

$$\delta I = \int_0^r \left(u'' + c^2 u\right) \delta u dx = 0$$

if $u(x)$ satisfies the Euler equation

$$u'' + c^2 u = 0, \quad u(0) = 0, u(r) = A. \tag{4.15}$$

The stationary solution $u(x)$ is

$$u(x) = \left(\frac{A}{\sin(cr)}\right) \sin(cx)$$

The Weierstrass test is satisfied, because the dependence of the Lagrangian on the derivative u' is convex, $\frac{\partial L}{\partial^2 u'^2} = c^2$.

The second variation equals

$$\delta^2 I = \int_0^r \left(\frac{1}{2} (\delta u')^2 - \frac{c^2}{2} (\delta u)^2 \right) dx$$

Since the ends of the trajectory are fixed, the variation δu satisfies homogeneous conditions $\delta u(0) = \delta u(r) = 0$. Let us choose the variation as follow:

$$\delta u = \begin{cases} \epsilon x(a-x), \ 0 \le x \le a \\ 0 \qquad\qquad\qquad x > a \end{cases}$$

where the interval of variation $[0, a]$ is not greater that $[0, r]$, $a \le r$. Computing the second variation, we obtain

$$\delta^2 I(a) = \frac{\epsilon^2}{60} a^3 (c^2 a^2 - 10), \quad a \le r$$

If second variation $\delta^2 I(a)$ is negative, $\delta^2 I(a) < 0$ the stationary solution does not correspond to the minimum of I. The second variation of the chosen type depends on a and $\delta^2 I$ is maximal when $a = r$. This maximum is negative when

$$r > r_{\text{crit}} = \frac{\sqrt{10}}{c}$$

We conclude that the stationary solution does not correspond to the minimum of I if the length of the trajectory is larger than r_{crit}. If the length is smaller than r_{crit}, the situation is inconclusive. We do not know at this point if it still possible to choose another type of variation different from considered here and disprove the optimality of the stationary solution.

4.2.2 Jacobi variation

The Jacobi necessary condition examines the optimality of "long" trajectories. It complements the Weierstrass test that investigates stability of a Lagrangian to strong localized variations. Jacobi condition tries to disprove optimality of a stationary trajectory by testing stability of the Lagrangian against nonlocal variations with small magnitude. This condition is stronger than the sufficient condition for the weak minimum.

Assume that a trajectory $u(x)$ satisfies the stationary condition and Weierstrass condition but does not satisfy the sufficient conditions for weak minimum, that is $Q(u, u')$ is not positively defined,

$$S(u, u') = 0, \quad \frac{\partial^2 F}{\partial (u')^2} > 0, \quad Q(u, u') \not> 0$$

To derive Jacobi condition, we apply again an infinitesimal nonlocal variation: $\delta u = O(\epsilon) \ll 1$ and $\delta u' = O(\epsilon) \ll 1$ and examine the expression (4.13) for the second variation. Notice that we denote the upper limit of integration in (4.13) by r; we are testing the stability of the trajectory depending on its length.

When a nonlocal "shallow" variation is applied, the increment increases because of assumed positivity of $\frac{\partial^2 F}{\partial (u')^2}$ and decreases because of assumed nonpositivity of the matrix Q. Depending on the length r of the interval of integration and of chosen form of the variation δu, one of these effects prevails. If the second effect is stronger, the extremal fails the test and is nonoptimal.

Jacobi conditions asks for the choice of the "best shape" δu of the variation. The expression (4.13) itself is a variational problem for δu which we rename as v; the Lagrangian is quadratic of v and v' and the coefficients are functions of x determined by the stationary trajectory $u(x)$:

$$\delta I = \int_0^r \left[A v^2 + 2B\, v\, v' + C (v')^2 \right] dx, \quad v(0) = v(r) = 0 \qquad (4.16)$$

where

$$A = \frac{\partial^2 F}{\partial u^2}, \quad B = \frac{\partial^2 F}{\partial u \partial u'}, \quad C = \frac{\partial^2 F}{\partial (u')^2}$$

are computed at the stationary trajectory u. The problem (4.16) is the variational problem for the unknown variation v. Its Euler equation is a solution to Storm-Liouville problem:

$$\frac{d}{dx}(Cv' + Bv) - Av = 0, \quad v(0) = v(r_{\text{conj}}) = 0 \quad \text{if } r < r_{\text{conj}} \qquad (4.17)$$

with boundary conditions $v(0) = v(r) = 0$. The point r_{conj} is called a conjugate point to the end of the interval. The problem is homogeneous: If $v(x)$ is a solution and c is a real number, $cv(x)$ is also a solution.

Jacobi condition is satisfied if the interval does not contain conjugate points, that is there is no nontrivial solutions to (4.17) on any subinterval of $[0, r_{\text{conj}}]$, that is if there are no nontrivial solutions of (4.17) with boundary conditions $v(r) = v(r_{\text{conj}}) = 0$ where $0 \leq r_{\text{conj}} \leq r$.

If this condition is violated, than there exist a family of trajectories

$$u(x) \begin{cases} u_0 + v & \text{if } \ x \in [0, r_{\text{conj}}] \\ u_0 & \text{if } \ x \in [r_{\text{conj}}, r] \end{cases}$$

that deliver the same value of the cost. Indeed, v is defined up to a multiplier: If v is a solution, αv is a solution too. These trajectories have discontinuous derivative at the points r_1 and r_2 which leads to a contradiction to the Weierstrass-Erdman condition that does not allow a broken extremal at these points.

Examples

Example 4.2.3 (Nonexistence of the minimizer: Blow up) Consider again the problem in example 4.2.2

$$I = \min_u \int_0^r \left(\frac{1}{2}(u')^2 - \frac{c^2}{2} u^2 \right) dx \quad u(0) = 0; \quad u(r) = A$$

The stationary trajectory and the second variation are give by formulas (4.12) and (4.12), respectively.

Instead of arbitrary choosing the second variation, we now choose it as a solution to the homogeneous problem (4.17) for $v = \delta u$

$$v'' + c^2 v = 0, \quad u(0) = 0, u(r_{\text{conj}}) = 0, \quad r_{\text{conj}} \leq r \tag{4.18}$$

This problem has a nontrivial solution $v = \epsilon \sin(cx)$ if the length of the interval is large enough to satisfy homogeneous condition of the right end, $cr_{\text{conj}} = \pi$ or

$$r \geq r(_{\text{conj}}) = \frac{\pi}{c}$$

The second variation $\delta^2 I$is negative when r is large,

$$\delta^2 I \leq \frac{1}{r}\epsilon^2 \left(\frac{\pi^2}{r^2} - c^2\right) < 0 \quad \text{if } r > \frac{\pi}{c}$$

which shows that the a stationary solution is not a minimizer.

To clarify the phenomenon, let us compute the stationary solution from the Euler equation (4.15). We have

$$u(x) = \left(\frac{A}{\sin(cr)}\right)\sin(cx) \quad \text{and } I(u) = \frac{A^2}{\sin^2(cr)}\left(\frac{\pi^2}{r^2} - c^2\right)$$

When r increases approaching the value $\frac{c}{\pi}$, the magnitude of the stationary solution indefinitely grows, and the cost indefinitely decreases:

$$\lim_{r \to \frac{c}{\pi} - 0} I(u) = -\infty$$

Obviously, the solution of the Euler equation that corresponds to finite $I(u)$ when $r > \frac{\pi}{c}$ is not a minimizer.

Remark 4.2.1 Comparing this result with the result in Example (4.2.3), we see that the optimal choice of variation improved the result at only 0.65%.

4.2.3 Does Nature minimize action?

The next example deals with a system of multiple degrees of freedom.

Consider the variational problem with the Lagrangian

$$L = \sum_{i=1}^{n} \frac{1}{2} m {u_i'}^2 - \frac{1}{2} C (u_i - u_{i-1})^2, \quad u(0) = u_0$$

We will see later in Chapter **??** that this Lagrangian describes the *action* of a chain of particles with masses m connected by springs with constant C. The second variation

$$\delta^2 L = \sum_{i=1}^{n} \frac{1}{2} m \dot{v_i}^2 - \frac{1}{2} C (v_i - v_{i-1})^2, \quad v_0 = 0, \quad v_n = 0$$

corresponds to the Euler equation – the eigenvalue problem

$$m\ddot{V} = \frac{C}{m} A V$$

where $V = v_1(t), \ldots, v_n(t)$ and

$$A = \begin{pmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & \ldots & 0 \\ 0 & 1 & -2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -2 \end{pmatrix}.$$

The problem has a solution – vector $v(t)$

$$v(t) = \sum \alpha_k v_k \sin \omega_k t \quad v(0) = v(T_{\text{conj}}) = 0, \quad T_{\text{conj}} \leq T$$

where v_k are the eigenvectors, α are coefficients found from initial conditions, and ω_k are the square roots of eigenvalues of the matrix A. Solving the characteristic equation for eigenvalues $\det(A - \omega^2 I) = 0$ we find that these eigenvalues are

$$\omega_k = 2\sqrt{\frac{C}{m}} \sin^2 \left(\sqrt{\frac{C}{m}} \frac{\pi k}{n} \right), \quad k = 1, \ldots n$$

The Jacobi condition is violated if $v(t)$ is consistent with the homogeneous initial and final conditions that is if the time interval is short enough. Namely, The condition is violated when the duration T is larger than

$$T \geq \frac{\pi}{\max(\omega_k)} \approx 2\pi \sqrt{\frac{m}{C}}$$

.

The continuous limit of the chain with the masses is achieved when the number N of notes indefinitely growth and their mass decreases correspondingly as $m(N) = \frac{m(0)}{N}$, and the stiffness of one link growth as $C(N) = C(0)N$ as it become N times shorter. Correspondingly,

$$\sqrt{\frac{C(N)}{m(N)}} = N\sqrt{\frac{C(0)}{m(0)}}$$

and the maximal eigenvalue ω_N tends to infinity as $N \to \infty$. This implies that the action J of the continuous system is not minimized at any time interval T.

What is minimized in classical mechanics? The action $L = T - V$ does not satisfy Jacobi condition because kinetic and potential energies, both convex functions or q and $\dot{q}$, enter the action with different signs. Therefore the action is not convex function of both q and $\dot{q}$. If would become convex, if we could change the sign of the kinetic energy. Formally, we may replace time t with the imaginary variable $t = i\tau$ and use the second-order homogeneity of T:

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}^T R(q)\dot{q} = -{q'_\tau}^T R(q) q'_\tau$$

The Lagrangian, considered as a function of q and q'_τ instead of q and $\dot{q}$, become a negative of a convex function if potential energy and $R(q)$ are convex. It become equal to the first integral: the energy

$$L(q, q'_\tau) = -{q'_\tau}^T R(q) q'_\tau - V(q)$$

The maximal of the variational problem

$$J = \max_{q(\tau)} \int_{t_0}^{t} L(q, q'_\tau) d\tau$$

does exist. The Euler equations are

They define a function $q(\tau)$. The true solution is $q(t) = \Re q(i\tau)$.

4.2.4 Overview: Norms in functional space

Calculus of variation studies increment of a functional at close-by curves. The answer to the question whether or not two curves are close to each other, depends on definition of closeness. This question is studied in theory of topological spaces. Unlike the distance between two points in finite-dimensional Euclidian space, the same two curves can be considered to be infinitesimally close or far parted depending of the meaning of "distance." The variational tests examine the stability of the stationary solutions to small perturbations; different tests differently define the smallness of perturbation.

In calculus of variations, there are three mostly used criteria to measure the closeness of two differentiable functions $f_1(x)$ and $f_2(x)$: The norm $\mathcal{N}_1$ of difference $\delta f(x) = f_1(x) - f_2(x)$ in the values of functions

$$\mathcal{N}_1(\delta f) = \max_{x \in (0,1)} |\delta f(x)|$$

the norm $\mathcal{N}_2$ of difference of their derivatives,

$$\mathcal{N}_2(\delta f) = \max_{x \in (0,1)} |\delta f'(x)|$$

and the length $\mathcal{N}_3$ of the interval on which these functions are different

$$\mathcal{N}_3(\delta f) = \Delta \quad \text{if} \;\; \delta f(x) = 0 \; \forall x \notin [x, x + \Delta]$$

None of variational tests guarantees the global optimality of the tested trajectory, only local minimum; at the other hand, these tests are simple enough to be applied to practically interesting problems. The *local minimum* satisfies the inequality

$$I(u) \le I(u + \delta u) \quad \forall \delta u : \; \mathcal{N}(\delta u(x)) < \varepsilon$$

where ε is infinitesimally small and $\mathcal{N}$ is a norm. The definition of what is local minimum depends on the above definitions of the norm $\mathcal{N}$.

If the perturbation is small in the following sense

$$\mathcal{N}_{\text{Legendre}}(\delta u) = \mathcal{N}_1(\delta u) + \mathcal{N}_2(\delta u) + \mathcal{N}_3(\delta u) < \varepsilon$$

the Legendre text is satisfied. The test assumes that the compared functions and their derivatives are close everywhere, and they are identical outside of an infinitesimal interval.

The Weierstrass text assumes that the compared functions are close everywhere, and they are identical outside of an infinitesimal interval, but their derivatives are not close in that infinitesimal interval of variation:

$$\mathcal{N}_{\text{Weierstrass}}(\delta u) = \mathcal{N}_1(\delta u) + \mathcal{N}_3(\delta u) < \varepsilon, \quad \mathcal{N}_2(\delta u) \text{ is arbitrary.}$$

If the objective functional satisfy the Weierstrass test we say that the extremal $u(x)$ realizes a *strong local minimum.* The Weierstrass test is stronger than the Legendre test.

The Jacobi test (see below, Section 4.2.2) assumes that

$$\mathcal{N}_{\text{Jacobi}}(\delta u) = \mathcal{N}_1(\delta u) + \mathcal{N}_2(\delta u) < \varepsilon, \quad \mathcal{N}_3(\delta u) \text{ is arbitrary}$$

that is the compared functions and their derivatives are close everywhere, but the variation is not localized. The Jacobi test is stronger than the Legendre test. If Jacobi test is satisfied we say that the extremal $u(x)$ realizes a *weak local minimum* (not to be confused with minimum of weakly convergent sequence or with minimum for localized variations). Neither Weierstrass and Jacobi tests is stronger than the other: They test the stationary trajectory from different angles.

Chapter 5

Constrained problems

5.1 Constrained minimum in finite-dimensional problems

5.1.1 Lagrangre Multiplyers method

Consider first a finite-dimensional problem of constrained minimum. Namely, we find the condition of the minimum:

$$\min_x f(x), \quad x \in R^n, \quad f \in C_2(R^n) \tag{5.1}$$

assuming that p constraints are applied

$$g_i(x_1, \dots x_n) = 0 \quad i = 1, \dots p \tag{5.2}$$

or in the vector form

$$\boldsymbol{g}(\boldsymbol{x}) = 0$$

where $\boldsymbol{g}$ and $\boldsymbol{x}$ are p- and n-dimensional vectors, respectively.

To find minimum, we add the constraints with the Lagrange multipliers $\boldsymbol{\mu} = (\mu_1, \dots \mu_p)$ and end up with the problem

$$\min_{\boldsymbol{x}} f(x) + \sum_i^p \mu_i g_i(x)$$

The stationary conditions become:

$$\frac{\partial f}{\partial x_k} + \sum_i^p \mu_i \frac{\partial g_i}{\partial x_k} = 0, \quad k = 1, \dots, n$$

or, in the vector form

$$\frac{\partial f}{\partial \boldsymbol{x}} + W \cdot \boldsymbol{\mu} = 0 \tag{5.3}$$

where the $p \times n$ Jacobian matrix W is

$$W = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} \quad \text{or, by elements, } W_{nm} = \frac{\partial g_n}{\partial x_m}$$

The system (5.3) together with the constraints (5.2) forms a system of $n+p$ equations for $n+p$ unknowns: Components of the vectors $\boldsymbol{x}$ and $\boldsymbol{\mu}$.

Example Consider the problem

$$J = \min_x \sum_i A_i^2 x_i$$

with the constraint

$$\sum_i \frac{1}{x_i - k} = \frac{1}{c}.$$

We rewrite it in the form:

$$J_a = \min_x \sum_i A_i^2 x_i + \lambda \left(\sum_i \frac{1}{x_i - k} - \frac{1}{c} \right).$$

From the condition $\frac{\partial J_a}{\partial \boldsymbol{x}} = 0$ we obtain

$$A_i^2 - \frac{\lambda}{(x_i - k)^2} = 0, \quad i = 1, \dots, n.$$

Then we find

$$\frac{1}{x_i - k} = \frac{|A_i|}{\sqrt{\lambda}}$$

and compose the equation for λ

$$\frac{1}{c} = \sum_i \frac{1}{x_i - k} = \frac{1}{\sqrt{\lambda}} \sum_i |A_i|$$

Now we find λ, the minimizer x_i

$$\sqrt{\lambda} = c \sum_i |A_i|, \quad x_i = k + \frac{\sqrt{\lambda}}{|A_i|},$$

and the value of the minimizing function J:

$$J = k \sum_i A_i^2 + c \left(\sum_i |A_i| \right)^2$$

5.1.2 How does it work? (Min-max approach)

Consider again the finite-dimensional minimization problem

$$\min_{x_1,\dots x_n} F(x_1,\dots x_n) \tag{5.4}$$

subject to one constraint

$$g(x_1,\dots x_n) = 0 \tag{5.5}$$

and assume that there exist solutions to (5.5) in the neighborhood of the minimal point.

It is easy to see that the described constrained problem is equivalent to the unconstrained problem

$$L_* = \min_{x_1,\dots x_n} \max_{\lambda} \left(F(x_1,\dots x_n) + \lambda g(x_1,\dots x_n)\right) \tag{5.6}$$

Indeed, the inner maximization gives

$$\max_{\lambda} \lambda g(x_1,\dots x_n) = \begin{cases} \infty & \text{if } g \neq 0 \\ 0 & \text{if } g = 0 \end{cases}$$

because λ can be made arbitrary large or arbitrary small. This possibility forces us to choose such $\boldsymbol{x}$ that delivers equality in (5.5), otherwise the cost of the problem (5.6) would be infinite (recall that x "wants" to minimize L_*). At the other hand, the constrained problem (5.4)-(5.5) does not change its value if zero $g = 0$ is added to it. Thereby the problem (5.4) and (5.5) is equivalent to (5.6).

If we are able to interchange the sequence of the two extremal operations in (5.6), we would arrive at the problem for the *augmented Lagrangian* L

$$L(\boldsymbol{x},\lambda) = \max_{\lambda} \min_{x_1,\dots x_n} \left(F(x_1,\dots x_n) + \lambda g(x_1,\dots x_n)\right) \tag{5.7}$$

Remark 5.1.1 Such interchange preserves the problems cost if $F(x_1,\dots x_n) + \lambda g(x_1,\dots x_n)$ is a convex function of $x_1,\dots x_n$; in this case $L = L_*$. In a general case, we arrive at an inequality $L \leq L_*$ (see the min-max theorem)

The extended Lagrangian L depends on $n+1$ variables. The stationary point corresponds to a solution to a system

$$\frac{\partial L}{\partial x_k} = \frac{\partial F}{\partial x_k} + \lambda \frac{\partial g}{\partial x_k} = 0, \quad k = 1,\dots n, \tag{5.8}$$

$$\frac{\partial L}{\partial \lambda} = g = 0 \tag{5.9}$$

The procedure is easily generalized for a case of several constrains. In this case, we add each constraint with its own Lagrange multiplier to the minimizing function and arrive at expression (5.3)

5.1.3 Excluding Lagrange multipliers

We can exclude the multipliers $\boldsymbol{\mu}$ from the system (5.3) as follows:

1. Multiply (5.3) by W^T:

$$G^T \frac{\partial f}{\partial \boldsymbol{x}} + W^T W \cdot \boldsymbol{\mu} = 0, \tag{5.10}$$

   Assume that constraints are independent or that $p \times p$ matrix $W^T W$ is nonsingular.

2. Find $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} = -(W^T W)^{-1} W^T \frac{\partial f}{\partial \boldsymbol{x}},$$

   (Notice that $W^T W$ is a nonnegative symmetric $p \times p$ matrix. It is invertible if the gradients $(g_i)\boldsymbol{x}$ are linearly independent.)

3. Substitute the obtained expression for $\boldsymbol{\mu}$ into (5.3) and obtain:

$$(I - W(W^T W)^{-1} W^T) \frac{\partial f}{\partial \boldsymbol{x}} = 0 \tag{5.11}$$

Notice that the rank of the matrix $W(W^T W)^{-1} W^T$ is equal to p; it has p eigenvalues equal to one and $n-p$ eigenvalues equal to zero. Therefore the rank of $I - W(W^T W)^{-1} W^T$ is equal to $n-p$, and the system (5.11) produces $n-p$ independent boundary conditions. The remaining p conditions are given by (5.2): $g_i = 0, \ i = 1, \dots p$. We consider several special cases.

$\boldsymbol{n}$ constraints Suppose that we assign n independent constraints. They define vector $\boldsymbol{x}$ and no additional condition is needed. Let us see what happens with our formula (5.11) in this case. The rank of the matrix $W(W^T W)^{-1} W^T$ is equal to n, (W^{-1} exists) therefore this matrix-projector is equal to I:

$$W(W^T W)^{-1} W^T = I$$

and the equation (5.11) becomes a trivial identity. No new condition is produced by (5.11) in this case, as it should be. The set of admissible values of $\boldsymbol{x}$ shrinks to the point and it is completely defined by the n equations $\boldsymbol{g}(\boldsymbol{x}) = 0$.

One constraint Another special case occurs if only one constraint is imposed; in this case $p = 1$ and the Lagrange multiplier $\boldsymbol{\mu}$ becomes a scalar and the conditions (5.3) have the form:

$$\frac{\partial f}{\partial x_i} + \mu \frac{\partial g}{\partial x_i} = 0 \quad i = 1, \dots n$$

Solving for μ and excluding it, we obtain $n-1$ stationary conditions:

$$\frac{\partial f}{\partial x_1} : \frac{\partial g}{\partial x_1} = \dots = \frac{\partial f}{\partial x_n} : \frac{\partial g}{\partial x_n} \tag{5.12}$$

Let us find how does this condition follow from the system (5.11). This time, W is a $1 \times n$ matrix, or a vector. We have:

$$\text{rank } W(W^TW)^{-1}W^T = 1, \quad \text{rank}(I - W(W^TW)^{-1}W^T) = n - 1$$

Matrix $I - W(W^TW)^{-1}W^T$ has $n-1$ eigenvalues equal to one and one zero eigenvalue that corresponds to the eigenvector W. At the other hand, optimality condition (5.11) states that the vector $\frac{\partial f}{\partial \boldsymbol{x}}$ (if it is not equal to zero) lies the null-space of the matrix $I - W(W^TW)^{-1}W^T$ that is the vectors $\frac{\partial f}{\partial \boldsymbol{x}}$ and W are parallel. Equation (5.12) expresses parallelism of these two vectors.

Exercise Consider minimization of a quadratic function (norm)

$$F = \boldsymbol{x}^T A \boldsymbol{x}$$

Subject to linear constraints

$$Bx = \beta$$

Derive the formulas for the minimal point.

5.1.4 Finite-dimensional variational problem revisited

Consider the optimization problem for a finite-difference system of equations

$$J = \min_{y_1, \ldots, y_N} \sum_i^N f_i(y_i, z_i)$$

where $f_1, \ldots, f_N$ are given value of a function f, $y_1, \ldots, y_N$ is the N-dimensional vector of unknowns, and z_i $i = 2, \ldots, N$ are the finite differences of y_i:

$$z_i = \text{Diff}(y_i) \quad \text{where } \text{Diff}(y_i) = \frac{1}{\Delta}(y_i - y_{i-1}), \quad i = 1, \ldots, N \tag{5.13}$$

Assume that the boundary values y_1 and y_n are given and take (5.13) as constraints. Using Lagrange multiplies $\mu_1, \ldots, \mu_N$ we pass to the augmented function

$$J_a = \min_{y_1, \ldots, y_N;\ z_1, \ldots, z_N} \sum_i^N \left[f_i(y_i, z_i) + \mu_i \left(z_i - \frac{1}{\Delta}(y_i - y_{i-1}) \right) \right]$$

The necessary conditions are:

$$\frac{\partial J_a}{\partial y_i} = \frac{\partial f_i}{\partial y_i} + \frac{1}{\Delta}(-\mu_i + \mu_{i+1}) = 0 \quad 2 = 1, \ldots, N-1$$

and

$$\frac{\partial J_a}{\partial z_i} = \frac{\partial f_i}{\partial z_i} + \mu_i = 0 \quad i = 2, \ldots, N-1$$

Excluding μ_i from the last equation and substituting their values into the previous one, we obtain the conditions:

$$\frac{\partial J_a}{\partial y_i} = \frac{\partial f_i}{\partial y_i} + \frac{1}{\Delta}\left(\frac{\partial f_i}{\partial z_i} - \frac{\partial f_{i+1}}{\partial z_{i+1}}\right) = 0 \quad i = 2, \ldots, N-1$$

or, recalling the definition of the Diff -operator,

$$\text{Diff}\left(\frac{\partial f_{i+1}}{\partial z_{i+1}}\right) - \frac{\partial f_i}{\partial y_i} = 0 \quad z_i = \text{Diff}(y_i) \tag{5.14}$$

One can see that the obtained necessary conditions have the form of the difference equation of second-order.

On the other hand, Diff-operator is an approximation of a derivative and the equation (5.14) is a finite-difference approximation of the Euler equation.

5.2 Isoperimetric problem

5.2.1 Stationarity conditions

Isoperimetric problem of the calculus of variations is

$$\min_u \int_a^b F(x, u, u')dx \quad \text{subject to} \quad \int_a^b G(x, u, u')dx = 0 \tag{5.15}$$

Applying the same procedure as in the finite-dimensional problem, we reformulate the problem using Lagrange multiplier λ:

$$\min_u \int_a^b \left[F(x, u, u') + \lambda G(x, u, u')\right] dx \tag{5.16}$$

To justify the approach, we may look on the finite-dimensional analog of the problem

$$\min_{u_i} \sum_{i=1}^N F_i(u_i, \text{Diff}(u_i)) \quad \text{subject to} \quad \sum_{i=1}^N G_i(u_i, \text{Diff}(u_i)) = 0$$

The Lagrange method is applicable to the last problem which becomes

$$\min_{u_i} \sum_{i=1}^N \left[F_i(u_i, \text{Diff}(u_i)) + \lambda G_i(u_i, \text{Diff}(u_i))\right].$$

Passing to the limit when $N \to \infty$ we arrive at (5.16).

The procedure of solution is as follows: First, we solve Euler equation for the problem(5.16)

$$\frac{d}{dx}\frac{\partial}{\partial u'}(F + \lambda G) - \frac{\partial}{\partial u}(F + \lambda G) = 0.$$

Keeping λ undefined and arrive at minimizer $u(x,\lambda)$ which depends on parameter λ. The equation

$$\int_a^b G(x, u(x,\lambda), u'(x,\lambda))dx = 0$$

defines this parameter.

Remark 5.2.1 The method assumes that the constraint is consistent with the variation: The variation must be performed upon a class of functions u that satisfy the constraint. Parameter λ has the meaning of the cost for violation of the constraint.

Of course, it is assumed that the constraint can be satisfied for all varied functions that are close to the optimal one. For example, the method is not applicable to the constraint

$$\int_a^b u^2 dx \leq 0$$

because this constraint allows for only one function $u = 0$ and will be violated at any varied trajectory.

5.2.2 Dido problem revisited

Let us apply the variational technique to Dido Problem discussed in Chapter **??**. It is required to maximize the area A between the OX axes and a positive curve $u(x)$

$$A = \int_a^b u dx \quad u(x) \geq 0 \forall x \in [a,b]$$

assuming that the length L of the curve is given

$$L = \int_a^b \sqrt{1+u'^2} dx$$

and that the beginning and the end of the curve belong to OX-axes: $u(a) = 0$ and $u(b) = 0$. Without lose of generality we assume that $a = 0$ and we have to find b.

The constrained problem has the form

$$J = A + \lambda L = \int_0^b \left(u + \lambda\sqrt{1+u'^2} \right) dx$$

where λ is the Lagrange multiplier.

The Euler equation for the extended Lagrangian is

$$1 - \lambda \frac{d}{dx} \left(\frac{u'}{\sqrt{1+u'^2}} \right)$$

Let us fix λ and find u as a function of x and λ. Integrating, we obtain

$$\lambda \frac{u'}{\sqrt{1+u'^2}} = x - C_1$$

where C_1 is a constant of integration. Solving for $u' = \frac{du}{dx}$, we rewrite the last equation as

$$du = \pm \frac{(x - C_1)dx}{\sqrt{\lambda^2 + (x - C_1)^2}},$$

integrate it:

$$u = \mp\sqrt{\lambda^2 + (x - C_1)^2} + C_2$$

and rewrite the result as

$$(x - C_1)^2 + (u - C_2)^2 = \lambda^2 \tag{5.17}$$

The extremal is a part of the circle. The constants C_1, C_2 and λ can be found from boundary conditions and the constraints.

To find the length b of the trajectory, we use the transversality condition (**??**):

$$u' \frac{\partial F}{\partial u'} - F|_{x=b} = -\frac{\lambda}{\sqrt{1+u'^2}} - u\Big|_{x=b,u(b)=0} = \frac{\lambda}{\sqrt{1+u'^2}}\Big|_{x=b} = 0$$

which gives $|u'(b)| = \infty$. We have shown that the optimal trajectory approaches OX-axis at the point b perpendicular to it. By symmetry, $|u'(a)| = \infty$ as well which means that the optimal trajectory is the semicircle of the radius λ, symmetric with respect to OX-axis. We find $\lambda = \frac{L}{\pi}$, $C_1 = a + \frac{L}{2\pi}$, and $C_2 = 0$.

5.2.3 Catenoid

The classical problem of the shape of a heavy chain (catenoid, from Latin *catena*) was considered by Euler ?? using a *variational principle*. It is postulated, that the equilibrium minimizes the potential energy W of the chain

$$W = \int_0^1 g\rho u \, ds = g\rho \int_0^1 u\sqrt{1 + (u')^2}dx$$

defined as the limit of the sum of vertical coordinates of the parts of the chain. Here, ρ is the density of the mass of the chain, ds is the element of its length, x and u are the horizontal and vertical coordinates, respectively. The length of the chain

$$L = \int_0^1 \sqrt{1 + (u')^2}dx$$

and the coordinates of the ends are fixed. Normalizing, we put $g\rho = 1$. Formally, the problem becomes

$$I = \min_{u(x)} \ (W(u) + \lambda L(u)), \quad W(u) + \lambda L(u) = \int_0^1 (u + \lambda)\sqrt{1 + (u')^2}dx$$

The Lagrangian is independent of x and therefore permits the first integral

$$(u+\lambda)\left(\frac{(u')^2}{\sqrt{1+(u')^2}}-\sqrt{1+(u')^2}\right)=C$$

that is simplified to

$$\frac{u+\lambda}{\sqrt{1+(u')^2}}=C.$$

We solve for u'

$$\frac{du}{dx}=\sqrt{\left(\frac{u+\lambda}{C}\right)^2-1}$$

integrate

$$x=\ln\left(\lambda+u+\sqrt{\left(\frac{u+\lambda}{C}\right)^2-1}\right)-\ln C+x_0$$

and find the extremal $u(x)$

$$u=-C\cosh\left(\frac{x-x_0}{C}\right)+\lambda$$

The equation – the catenoid – defines the shape of a chain; it also gave the name to the hyperbolic cosine.

5.3 General form of a variational functional

5.3.1 Reduction to isoperimetric problem

Lagrange method allows for reformulation of an extremal problem in a general form as a simplest variational problem. The minimizing functional can be the product, ratio, superposition of other differentiable function of integrals of the minimizer and its derivative. Consider the problem

$$J=\min_u \Phi(I_1,\ldots,I_n) \tag{5.18}$$

where

$$I_k(u)=\int_a^b F_k(x,u,u')dx \quad k=1,\ldots n \tag{5.19}$$

and Φ is a continuously differentiable function. Using Lagrange multipliers λ_1,λ_n, we transform the problem (5.18) to the form

$$J=\min_u \min_{I_1,\ldots,I_n} \max_{\lambda_1,\ldots\lambda_n}\left\{\Phi+\sum_{k=1}^n \lambda_k\left(I_k-\int_a^b F_k(x,u,u')dx\right)\right\}. \tag{5.20}$$

The stationarity conditions for (5.20) consist of n algebraic equations

$$\frac{\partial \Phi}{\partial I_k} + \lambda_i = 0 \tag{5.21}$$

and the differential equation – the Euler equation

$$S(\Psi, u) = 0$$

$$\left(\text{recall that } S(\Psi, u) = \frac{d}{dx}\frac{\partial \Psi}{\partial u'} - \frac{\partial \Psi}{\partial u}\right)$$

for the function

$$\Psi(u) = \sum_{k=1}^{n} \lambda_k F_k(x, u, u')$$

Together with the definitions (5.19) of I_k, this system enables us to determine the real parameters I_k and λ_k and the function $u(x)$. The Lagrange multipliers can be excluded from the previous expression using (5.21), then the remaining stationary condition becomes an integro-differential equation

$$S(\bar{\Psi}, u) = 0, \quad \bar{\Psi}(I_k, u) = \sum_{k=1}^{n} \frac{\partial \Phi}{\partial I_k} F_k(x, u, u') \tag{5.22}$$

Next examples illustrate the approach.

The product of integrals

Consider the problem

$$\min_u J(u), \quad J(u) = \left(\int_a^b \phi(x, u, u')dx\right)\left(\int_a^b \psi(x, u, u')dx\right).$$

We rewrite the minimizing quantity as

$$J(u) = I_1(u) I_2(u), \quad I_1(u) = \int_a^b \phi(x, u, u')dx, \quad I_2(u) = \int_a^b \psi(x, u, u')dx,$$

apply stationary condition (5.22), and obtain the condition

$$I_1 \delta I_2 + I_2 \delta I_1 = I_2(u) S(\phi(u), u) + I_1(u) S(\psi(u), u) = 0. \tag{5.23}$$

or

$$\left(\int_a^b \phi(x, u, u')dx\right)^{-1} S(\phi(u), u) + \left(\int_a^b \psi(x, u, u')dx\right)^{-1} S(\psi(u), u) = 0$$

The equation is nonlocal: Solution u at each point depends on its first and second derivatives and integrals of $\phi(x, u, u')$ and $\phi(x, u, u')$ over the whole interval $[a, b]$.

Example 5.3.1 Solve the problem

$$\min_u \left(\int_0^1 (u')^2 dx\right)\left(\int_0^1 (u+1)dx\right) \quad u(0)=0, \quad u(1)=a$$

We denote

$$I_1 = \int_0^1 (u')^2 dx, \quad I_2 = \int_0^1 (u+1)dx$$

and compute the Euler equation using (5.23)

$$I_2 u'' - I_1 = 0, \quad u(0)=0, \quad u(1)=a.$$

or

$$u'' - R = 0, \quad u(0)=0, \quad u(1)=a, \quad R = \frac{I_1}{I_2}$$

The integration gives

$$u(x) = \frac{1}{2}Rx^2 + \left(a - \frac{1}{2}R\right)x,$$

We obtain the solution that depends on R – the ratio of the integrals of two function of this solution. To find R, we substitute the expression for $u = u(R)$ into right-hand sides of I_1 and I_2,

$$I_1 = \frac{R^2}{12} + a^2, \quad I_2 = -\frac{R}{12} + \frac{1}{2}a + 1$$

compute the ratio, $\frac{I_1}{I_2} = R$ and obtain the equation for R,

$$R = \frac{R^2 + 12a^2}{R + 6a + 12}$$

Solving it, we find $R = \frac{1}{2}(3a + 6 \pm \sqrt{36 + 36a - 15a^2})$.

At this point, we do not know whether the solution correspond to minimum or maximum. This question is investigated later in Chapter 4.

The ratio of integrals

Consider the problem

$$\min_u J(u), \quad J(u) = \frac{\int_a^b \phi(x,u,u')dx}{\int_a^b \psi(x,u,u')dx}.$$

We rewrite it as

$$J = \frac{I_1}{I_2}, \quad I_1(u) = \int_a^b \phi(x,u,u')dx, \quad I_2(u) = \int_a^b \psi(x,u,u')dx, \qquad (5.24)$$

apply stationary condition (5.22), and obtain the condition

$$\frac{1}{I_2(u)} S(\phi(u), u) - \frac{I_1(u)}{I_2^2(u)} S(\psi(u), u) = 0.$$

Multiplying this equality by I_2 and using definition (5.24) of the goal functional, we bring the previous expression to the form

$$S(\phi, u) - J\, S(\psi, u) = S(\phi - J\psi, u) = 0$$

Observe that the stationarity condition depends on the cost J of the problem.

Example 5.3.2 Solve the problem

$$\min_u J(u), \quad J = \frac{\int_0^1 (u')^2 dx}{\int_0^1 (u-1) dx} \quad u(0) = 0, \quad u(1) = a$$

We compute the Euler equation

$$u'' - J = 0, \quad u(0) = 0, \quad u(1) = a.$$

where

$$R = \frac{I_1}{I_2}, \quad I_1 = \int_0^1 (u')^2 dx, \quad I_2 = \int_0^1 (u-1) dx$$

The integration gives

$$u(x) = \frac{1}{2} R x^2 + \left(a - \frac{1}{2} R\right) x,$$

We obtain the solution that depends on R – the ratio of the integrals of two function of this solution. To find R, we substitute the expression for $u = u(R)$ into right-hand sides of I_1 and I_2,

$$I_1 = \frac{R^2}{12} + a^2, \quad I_2 = -\frac{R}{12} + \frac{1}{2} a + 1$$

compute the ratio, $\frac{I_1}{I_2} = R$ and obtain the equation for R,

$$R = \frac{R^2 + 12a^2}{R + 6a + 12}$$

Solving it, we find $R = \frac{1}{2}(3a + 6 \pm \sqrt{36 + 36a - 15a^2})$.

At this point, we do not know whether the solution correspond to minimum or maximum. This question is investigated later in Chapter 4.

The examples will be given in the next section.

Superposition of integrals

Consider the problem

$$\min_u \int_a^b R\left(x, u, u', \int_a^b \phi(x, u, u')dx\right) dx$$

We introduce a new variable I

$$I = \int_a^b \phi(x, u, u')dx$$

and reformulate the problem as

$$\min_u \int_a^b \left[R(x, u, u', I) + \lambda\left(\phi(x, u, u') - \frac{I}{b-a}\right)\right] dx$$

where λ is the Lagrange multiplier. The stationarity conditions are:

$$S((R + \lambda\phi), u) = 0, \quad \frac{\partial R}{\partial I} - \frac{1}{b-a} = 0.$$

and the above definition of I.

Example 5.3.3 (Integral term in the Lagrangian) Consider the following extremal problem posed in "physical terms": Find the function $u(x)$ at the interval $[0, 1]$ that is has prescribed values at its ends,

$$u(0) = 1, \quad u(1) = 0, \tag{5.25}$$

has a smallest L_2-norm

$$\int_0^1 u'^2 dx$$

of the derivative u', and stays maximally close to its averaged over the interval $[0, x]$ value a,

$$a = \int_0^x u(t)dt \tag{5.26}$$

In order to formulate a mathematical extremal problem, we combine the two above requests on $u(x)$ the into one Lagrangian F equal to the weighted sum of them:

$$F = u'^2 + \alpha\left(u - \int_0^1 u(t)dt\right)^2, \quad u(0) = 1, \quad u(1) = 0$$

where $\alpha \geq 0$ is a weight coefficient that show the relative importance of the two criteria. Function $u(x)$ is a solution to the extremal problem

$$\min_{u(x),\ u(0)=1, u(1)=0} \int_0^1 F\left(u, u', \int_0^1 u(t)dt\right) dx \tag{5.27}$$

We end up with the variational problem with the Lagrangian that depends on the minimizer u, its derivative and its integral.

Remark 5.3.1 Formulating the problem, we could include boundary conditions into a minimized part of the functional instead of postulating them; in this case the problem would be

$$\min_{u(x)} \left\{ \int_0^1 F\left(u, u', \int_0^1 u(t)dt\right) dx + \beta_1 (u(0)-1)^2 + \beta_2 u(1)^2 \right\}$$

where $\beta_1 \geq 0$ and $\beta_2 \geq 0$ are the additional weight coefficients.

We bring the problem (5.27) to the form of the standard variational problem, accounting for the equality (5.26) with the Lagrange multiplier λ; the objective functional becomes

$$J = \int_0^1 \left(u'^2 + \alpha(u-a)^2\right) dx + \lambda \left(a - \int_0^1 u\, dx\right)$$

or

$$J = \int_0^1 \left(u'^2 + \alpha(u-a)^2 + \lambda(a-u)\right) dx$$

The parameter a and the function $u(x)$ are the unknowns. The stationary condition with respect to a is

$$\frac{\partial J}{\partial a} = \int_0^1 \left(-2\alpha(u-a) + \lambda\right) dx = 2\alpha a + \lambda - 2 \underbrace{\int_0^1 u\, dx}_{=a} = 0,$$

it allows for linking a and λ,

$$\lambda = 2(\alpha - 1)a.$$

The stationary condition with respect to $u(x)$ (Euler equation) is

$$2u'' - 2\alpha(u-a) - \lambda = 0$$

We exclude λ using the obtained expression for λ, and obtain

$$2u'' - 2\alpha u + a = 0 \tag{5.28}$$

The integro-differential system (5.26) and (5.28) with the boundary conditions (5.25) determines the minimizer.

To solve the system, we first solve (5.28) and (5.25) treating a as a parameter,

$$u(x) = \frac{a}{2\alpha} + A \sinh(\sqrt{\alpha} x) + B \cosh(\sqrt{\alpha} x)$$

where

$$A = \left(\frac{a}{2\alpha} - 1\right) \frac{\cosh(\sqrt{\alpha})}{\sinh(\sqrt{\alpha})}, \quad B = 1 - \frac{a}{2\alpha},$$

and substitute this solution into (5.26) obtaining the linear equation for the remaining unknown a. We have

$$u(x) = c_1(x) a + c_2(x)$$

where

$$c_1(x) = \frac{1}{2\alpha}\left(1 + \frac{\cosh(\sqrt{\alpha})}{\sinh(\sqrt{\alpha})}\sinh(\sqrt{\alpha}x) - \cosh(\sqrt{\alpha}x)\right)$$

and

$$c_2(x) = \left(\cosh(\sqrt{\alpha}x) - \frac{\cosh(\sqrt{\alpha})}{\sinh(\sqrt{\alpha})}\sinh(\sqrt{\alpha}x)\right)$$

and (5.26) becomes

$$a = a\int_0^1 c_1(x)dx + \int_0^1 c_2(x)dx$$

which implies

$$a = \frac{\int_0^1 c_2(x)dx}{\int_0^1 c_1(x)dx - 1}$$

The general procedure is similar: We always can rewrite a minimization problem in the standard form adding new variables (as the parameter c in the previous examples) and corresponding Lagrange multipliers.

Inequality in the isoperimetric condition Often, the isoperimetric constraint is given in the form of an inequality

$$\min_u \int_a^b F(x,u,u')dx \quad \text{subject to} \int_a^b G(x,u,u')dx \geq 0 \tag{5.29}$$

In this case, the additional condition $\lambda \geq 0$ is added to the Euler-Lagrange equations (**??**) according to the (**??**).

Remark 5.3.2 Sometimes, the replacement of an equality constraint with the corresponding inequality can help to determine the sign of the Lagrange multiplier. For example, consider the Dido problem, and replace the condition that the perimeter is fixed with the condition that the perimeter is smaller than or equal to a constant. Obviously, the maximal area corresponds to the maximal allowed perimeter and the constraint is always active. On the other hand, the problem with the inequality constraint requires positivity of the Lagrange multiplier; so we conclude that the multiplier is positive in both the modified and original problem.

5.3.2 Homogeneous functionals and Eigenvalue Problem

The next two problems are *homogeneous*: The functionals do not vary if the solution is multiplied by any number. Therefore, the solution is defined up to a constant multiplier.

The eigenvalue problem corresponds to the functional

$$I_1 = \min_u \frac{\int_0^1 (u')^2 dx}{\int_0^1 u^2 dx} \quad x(0) = x(1) = 0 \tag{5.30}$$

it can be compared with the problem:

$$I_2 = \min_u \frac{\int_0^1 (u')^2 dx}{\left(\int_0^1 u dx\right)^2} \quad x(0) = x(1) = 0 \tag{5.31}$$

Do these problem have nonzero solutions?

Consider the problem (5.30). Because the solution is defined up to a multiplier, we can normalize it assuming that

$$\int_0^1 u^2 dx = 1 \tag{5.32}$$

Then the problem takes the form

$$I_1 = \min_u \int_0^1 \left((u')^2 + \lambda u^2\right) dx \quad x(0) = x(1) = 0$$

where λ is the Lagrange multiplier by the normalization constraint (5.32). The Euler equation is

$$u'' - \lambda u = 0, \quad x(0) = x(1) = 0$$

This equation represents the eigenvalue problem. It has nonzero solutions u only if λ takes special values – the eigenvalues. These values are $\lambda_n = -(\pi n)^2$ where n is a nonzero integer; the corresponding solutions – the eigenfunctions u_n – are equal to $u_n(x) = C\sin(\pi n x)$. The constant C is determined from the normalization (5.32) as $C = \sqrt{2}$. The cost of the problem at a stationary solution u_n is

$$\int_0^1 (u_n')^2 \, dx = n^2\pi^2$$

The minimal cost I_1 corresponds to $n = 1$ and is equal to $I_1 = \pi^2$

The problem (5.31) is also homogeneous, and its solution u is defined up a multiplier. We reformulate the problem by normalizing the solution,

$$\int_0^1 u dx = 1.$$

The problem (5.31) becomes

$$\min_u \int_0^1 \left((u')^2 + \lambda u\right) dx \quad x(0) = x(1) = 0$$

where λ is the Lagrange multiplier by the normalization constraint.

The minimizer u satisfies the Euler equation

$$u'' - \frac{\lambda}{2} = 0, \quad x(0) = x(1) = 0$$

and is equal to $u = \frac{\lambda}{2}x(x-1)$. The constraint gives $\lambda = 12$ and the objective is

$$\int_0^1 (u')^2 dx = \int_0^1 (6 - 12x)^2 dx = 12$$

These two homogeneous variational problems correspond to different types of Euler equation. The equation for the problem (5.30) is homogeneous; it has either infinitely many solutions or no solutions depending on λ. It can select the stationary solution set but cannot select a solution inside the set: this is done by straight comparison of the objective functionals. The problem (5.31) leads to hon-homogeneous Euler equation that linearly depend on the constant λ of normalization. It has a unique solution if the normalization constant is fixed.

Homogeneous with a power functionals To complete the considerations, consider a larger class of homogeneous with a power p functionals, $I(qu) = q^p I(u)$ where $q > 0$ is an arbitrary positive number. For example function $I(x) = ax^4$ is homogeneous with the power four, because $I(qx) = aq^4x^4 = q^4 I(x)$. Here, $p \neq 1$ is a real number. for all u. For example, the functional can be equal to

$$J_3(u) = \frac{\int_0^1 (u')^2 dx}{\left|\int_0^1 u dx\right|^p}, \quad x(0) = x(1) = 0, \quad u \not\equiv 0 \tag{5.33}$$

which implies that it is homogeneous with the power $2 - p$, because $J_3(qu) = q^{2-p} J_3(u)$.

The minimization of such functionals leads to a trivial result: Either $\inf_u J_3 = 0$ or $\inf_u J_3 = -\infty$, because the positive factor q^p can be make arbitrarily large or small.

More exactly, if there exist u_0 such that $I(u_0) \leq 0$, than $\inf_u J_3 = -\infty$; the minimizing sequence consists of the terms $q_k u_0$ where the multipliers q_k are chosen so that $\lim q_k^p = \infty$.

If $I(u_0) \geq 0$ for all u_0, than $\inf_u J_3 = 0$; the minimizing sequence again consists of the terms $q_k u_0$ where the multipliers q_k are chosen so that $\lim q_k^p = 0$.

Remark 5.3.3 In the both cases, the minimizer itself does not exist but the minimizing sequence can be built. These problems are examples of variational problems without classical solution that satisfies Euler equation. Formally, the solution of problem (5.33) does not exist because the class of minimizers is open: It does not include $u \equiv 0$ and $u \equiv \infty$ one of which is the minimizer. We investigate the problems without classical solutions in Chapter **??**.

5.3.3 Constraints in boundary conditions

Constraints on the boundary, fixed interval Consider a variational problem (in standard notations) for a vector minimizer $\boldsymbol{u}$. If there are no constrains imposed on the end of the trajectory, the solution to the problem satisfies n natural boundary conditions

$$\delta \boldsymbol{u}(b) \cdot \left. \frac{\partial F}{\partial \boldsymbol{u}'} \right|_{x=b} = 0$$

(For definiteness, we consider here conditions on the right end, the others are clearly identical).

The vector minimizer of a variational problem may have some additional constraints posed at the end point of the optimal trajectory. Denote the boundary value of $u_i(b)$ by v_i The constraints are

$$\phi_i(v_1, \ldots v_n) = 0 \quad i = 1, \ldots, k; \quad k \le n$$

or in vector form,

$$\Phi(x, \boldsymbol{v}) = 0,$$

where Φ is the corresponding vector function. The minimizer satisfies these conditions and $n - k$ supplementary natural conditions that arrive from the minimization requirement. Here we derive these supplementary boundary conditions for the minimizer.

Let us add the constrains with a vector Lagrange multiplier $\boldsymbol{\lambda} = (\lambda_1, \ldots . \lambda_k)$ to the problem. The variation of $\boldsymbol{v} = \boldsymbol{u}(b)$ gives the conditions

$$\delta \boldsymbol{v} \cdot \left[\left. \frac{\partial F}{\partial \boldsymbol{u}'} \right|_{x=b, \boldsymbol{u}=\boldsymbol{v}} + \frac{\partial \Phi}{\partial \boldsymbol{v}} \boldsymbol{\lambda} \right] = 0$$

The vector in the square brackets must be zero because of arbitrariness of $\nu = \delta \boldsymbol{u}(b)$.

Next, we may exclude $\boldsymbol{\lambda}$ from the last equation (see the previous section 5.1.3):

$$\boldsymbol{\lambda} = - \left[\left(\frac{\partial \Phi}{\partial \boldsymbol{u}} \right)^T \frac{\partial \Phi}{\partial \boldsymbol{u}} \right]^{-1} \left. \frac{\partial F}{\partial \boldsymbol{u}'} \right|_{x=b, \boldsymbol{u}=\boldsymbol{v}} \tag{5.34}$$

and obtain the conditions

$$\left(I - \frac{\partial \Phi}{\partial \boldsymbol{u}}^T \left[\left(\frac{\partial \Phi}{\partial \boldsymbol{u}} \right)^T \frac{\partial \Phi}{\partial \boldsymbol{u}} \right]^{-1} \frac{\partial \Phi}{\partial \boldsymbol{u}} \right) \left. \frac{\partial F}{\partial \boldsymbol{u}'} \right|_{x=b, \boldsymbol{u}=\boldsymbol{v}} = 0 \tag{5.35}$$

The rank of the matrix in the parenthesis is equal to $n - k$. Together with k constrains, these conditions are the natural conditions for the variational problem.

Example

$$\min_{u_1, u_2} \int_a^b (u_1'^2 + u_2'^2 + u_3') dx, \quad u_1(b) + u_2(b) = 1, \; u_1(b) - u_3(b) = 1,$$

We compute

$$\frac{\partial F}{\partial \boldsymbol{u}'} = \begin{pmatrix} 2u_1 \\ 2u_2 \\ 1 \end{pmatrix}, \quad \frac{\partial \Phi}{\partial \boldsymbol{u}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(please continue..)

Free boundary with constraints Consider a general case when the constraints $\Phi(x, u) = 0$ are posed on the solution at the end point. Variation of these constrains results in the condition:

$$\delta\Phi(x,u)|_{x=b} = \frac{\partial \Phi}{\partial u}\delta u + \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u}u'\right)\delta x$$

Adding the constraints to the problem with Lagrange multiplier $\boldsymbol{\lambda}$, performing variation, and collecting terms proportional to δx, we obtain the condition at the unknown end point $x = b$

$$F(x, u, u') - \frac{\partial F}{\partial u'}u' + \boldsymbol{\lambda}^T \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u}u'\right) = 0$$

where $\boldsymbol{\lambda}$ is defined in (5.34). Together with $n - k$ conditions (5.35) and k constraints, they provide $n + 1$ equations for the unknowns $u_1(b), \ldots, u_n(b), b$.

5.4 Pointwise Constraints

5.4.1 Stationarity conditions

Consider a variational problem for a vector-valued minimizer $u = u_1, \ldots u_n$.

$$\min_u \int_a^b F(x, \boldsymbol{u}, \boldsymbol{u}')dx$$

Assume that the minimizer obeys certain constraint(algebraic or differential) in each point of any admissible trajectory,

$$G(x, \boldsymbol{u}, \boldsymbol{u}') = 0, \forall x \in (a, b) \tag{5.36}$$

The number of constraints is less than the number of minimizers. This way, we arrive at the constrained variational problem

$$\min_u \int_a^b F(x, \boldsymbol{u}, \boldsymbol{u}')dx \quad \text{subject to } G(x, \boldsymbol{u}, \boldsymbol{u}') = 0, \forall x \in (a, b) \tag{5.37}$$

As in the isoperimetric problem, we use the Lagrange multipliers method to account for the constrain. This time, the constraint must be enforced in every point of the trajectory, therefore the Lagrange multiplier becomes a function of x. To prove the method, it is enough to pass to finite-dimensional problem; after discretization, the constraint is replaced by the array of equations

$$G(x,u,u') = 0 \;\Rightarrow\; G_i(u_i, \mathrm{Diff}\,(u_i)) = 0, \quad i = 1, \dots N.$$

Each of this constrains, multiplied by its own Lagrange multiplier $\mu_1, \dots \mu_N$, must be added to the functional. The array of these multipliers converge to a function $\mu(x)$ when $N \to \infty$. The variational problem becomes

$$\min_u \int_a^b [F(x,u,u') + \mu(x) G(x,u,u')]\, dx \tag{5.38}$$

The necessary conditions are expressed in the form of differential constraints (5.36) and Euler equation:

$$G(x,u,u') = 0 \tag{5.39}$$

$$\left(\frac{d}{dx}\frac{\partial}{\partial u'}\frac{\partial}{\partial u}\right)(F + \mu G) = 0. \tag{5.40}$$

They define functions $u(x)$ and $\mu(x)$.

Algebraic constraints Notice that if the constraints are algebraic, $G = G(x,u)$, then (5.40) does not depend on μ' and is an algebraic relation for μ.

Consider the case of one constraint $G(x,u) = 0$. The Euler equations are

$$\frac{d}{dx}\frac{\partial F}{\partial u_k'} - \frac{\partial F}{\partial u_k} - \mu \frac{\partial G}{\partial u_k} = 0, \quad k = 1 \dots, n.$$

We may exclude $\mu = \mu(x)$ from the system and obtain $n-1$ equations

$$\left(\frac{\partial F}{\partial u_1} - \frac{d}{dx}\frac{\partial F}{\partial u_1'}\right)\left(\frac{\partial G}{\partial u_1}\right)^{-1} = \left(\frac{\partial F}{\partial u_k} - \frac{d}{dx}\frac{\partial F}{\partial u_k'}\right)\left(\frac{\partial G}{\partial u_k}\right)^{-1}, \quad k = 2, \dots, n$$

for $u_1, \dots u_n$; this system is supplemented with the constraint $G(x,u) = 0$.

The general case is considered similarly. Euler equation forms a linear system for vector-function μ; it can be excluded from the system, as it will be shown in following examples.

Example 5.4.1 (Euler equation revisited) As a first example, we derive Euler equation in a different manner: The minimization problem

$$\min_u \int_a^b F(x,u,v) dx \quad \text{subject to } v = u' \tag{5.41}$$

is obviously equivalent to the canonic variational problem.

Using Lagrange multiplier, we rewrite the problem as

$$\min_u \int_a^b (F(x,u,v) + \mu(u' - v))\, dx$$

Variation with respect to u, v gives, respectively,

$$\mu' - \frac{\partial F}{\partial u} = 0, \quad \mu + \frac{\partial F}{\partial v} = 0$$

(the term $\mu u'$ is transformed by integration by parts). We exclude μ by differentiation of the second equation and subtraction of the first one:

$$\left(\frac{\partial F}{\partial v}\right)' - \frac{\partial F}{\partial u} = 0$$

Accounting for the constraint $v = u'$ we arrive at Euler equation.

Geodesics as constrained problem We return to the problem of geodesics – the shortest path on a surface between two points at this surface. Here we will develop a general approach to the problem without assumptions that the local coordinates and the metric is introduced on the surface. We simply assume that the surface is parameterized as

$$\psi(x_1, x_2, x_3) = 0 \tag{5.42}$$

where x_1, x_2, x_3 are Cartesian coordinates. The distance D along a path $x(t), y(t), z(t)$ is

$$D = \int_0^1 \sqrt{x_1'(t)^2 + x_2'(t)^2 + x_3'(t)^2} dt$$

The extended Lagrangian is

$$F = \sqrt{x_1'(t)^2 + x_2'(t)^2 + x_3'(t)^2} + \mu(t)\psi(x_1, x_2, x_3) = 0$$

where $\mu(t0$ is the Lagrange multiplier. Euler equations are

$$\frac{d}{dt}\frac{x_i'}{\sqrt{x_i'(t)^2 + x_2'(t)^2 + x_3'(t)^2}} - \mu\frac{\partial \psi}{\partial x_i} = 0, \quad i = 1, 2, 3.$$

Excluding μ, we obtain two equalities

$$\left(\frac{\partial \psi}{\partial x_i}\right)^{-1} \frac{d}{dt}\left(\frac{x_i'}{\sqrt{x_i'(t)^2 + x_2'(t)^2 + x_3'(t)^2}}\right) = \mu(t) \quad i = 1, 2, 3$$

which together with equation (5.42) for the surface determine the optimal trajectory: the geodesic.

5.4.2 Constraints in the form of differential equations

The same idea of constrained variational problem can be used to account for the differential equations of the motion as constraints

$$g(u, u') = 0 \forall x \in [0, 1]. \tag{5.43}$$

This idea is fully exploited in the control theory, (see below, Section ??). The formal scheme is as in the previous case, but this time the derivatives of the Lagrange multipliers participate in the Euler equation:

$$\frac{dd}{dx}\left(\frac{\partial F}{\partial u'}+\lambda\frac{\partial g}{\partial u'}\right)-\frac{\partial F}{\partial u}-\frac{\partial g}{\partial u'}$$

that should be solved together with (5.43) to determine u and λ. Here we illustrate it by an example.

Example 5.4.2 (Antiderivative in Lagrangian) Consider an extremal problem similar to Example 5.3.3: Find the function $u(x)$ at the interval $[0,1]$ that is equal to one at its ends, $u(0)=1$ and $u(1)=1$, has a smallest L_2-norm

$$\int_0^1 u'^2 dx$$

of the derivative u', and at each instance x stays maximally close to its accumulated over the interval $[0,x]$ value v(x),

$$v(x)=\int_0^x u(t)dt \tag{5.44}$$

Combine the two above requests on $u(x)$, we form the Lagrangian F equal to the weighted sum of them:

$$F=u'^2+\alpha\left(u-\int_0^x u(t)dt\right)^2$$

where $\alpha\geq 0$ is a weight coefficient. Function $u(x)$ is a solution to the extremal problem

$$\min_{u(x),\ u(0)=u(1)=1}\int_0^1 F\left(u,u',\int_0^x u(t)dt\right)dx$$

We end up with the variational problem with the Lagrangian that depends on the minimizer u, its derivative, and its antiderivative.

To deal with this problem, we first reformulate it as a standard variational problem. Differentiation of the definitive equation (5.44) replaces it with the equivalent differential equation for v,

$$v'=u,\quad v(0)=0$$

and allows to rewrite the Lagrangian in the form

$$F_{\text{ext}}(u,v)=u'^2+\alpha\,(u-v)^2+\lambda(v'-u)$$

where $\lambda=\lambda(x)$ is the Lagrange multiplier by the differential constrain $v'=u$.

The Euler equations express the stationarity with respect to variations of u and v,

$$\begin{aligned}\frac{d}{dx}\frac{\partial F_{\text{ext}}}{\partial u'}-\frac{\partial F_{\text{ext}}}{\partial u} &= 2u''-2\alpha\,(u-v)+\lambda=0\\ \frac{d}{dx}\frac{\partial F_{\text{ext}}}{\partial v'}-\frac{\partial F_{\text{ext}}}{\partial v} &= \lambda'+2\alpha\,(u-v)=0;\end{aligned}$$

the natural boundary condition is $\lambda(1) = 0$

To exclude λ we differentiate the first equation and subtract the second:

$$2u''' - 2\alpha\,(u' - v') - 2\alpha\,(u - v) = 0$$

Now, use the definition of v: $u = v'$ and exclude u and its derivatives, obtaining a single equation for $v(x)$,

$$v^{IV} - \alpha v'' + \alpha v = 0$$

The boundary conditions are

$$v(0) = 0, \quad v'(0) = 1, \quad v'(1) = 1, \quad \lambda(1) = 2(v'''(1) - \alpha v'(1) + \alpha v(1)) = 0$$

The solution to this linear problem is obtained by a standard procedure.

Problem 5.4.1 Derive the equations for the case when the boundary terms are not prescribed by included into Lagrangian. Discuss the difference.

Problem 5.4.2 Replace the accumulated value by average value and obtain differential equations and boundary conditions

Sailing boat

Consider the problem: How to use the minimal resources to sail to a proper destination. First, let us do the modelling. The equations of the boat in the water are

$$m\ddot{x} + \gamma\dot{x} = f(t)$$

where x is the coordinate of the boat, m is its mass, γ is the dissipation, and $f(t)$ is the time-dependent driving force that depends on the used amount of fuel

$$|f| = r^q.$$

It is required to bring the boat to the moorage $x(T) = P$ from the moorage $x(0) = 0$ in the given time T; the speed in the beginning and in the end must be zero, $\dot{x}(0) = \dot{x}(T) = 0$.

The total amount R of the fuel

$$R = \int_0^T r(t)dt \tag{5.45}$$

must be minimized:

Remark 5.4.1 In the modelling, it was assumed that the boat is moving straight from the start to the destination and the forward and backward acceleration require the same amount of fuel.

We formulate the variational problem for the unknown fuel consumption rate $r(t)$ and the boat's speed $v(t) = \dot{r}$ subject to differentia constraint

$$m\dot{v} + \gamma v = r^q, \tag{5.46}$$

boundary conditions

$$v(0) = v(T) = 0,$$

and the integral constraint

$$\int_0^T v(t) = L. \tag{5.47}$$

Accounting for the constrains (5.46) and (5.47) by Lagrange multiplier $\lambda = \lambda(x)$ and $\mu =$ constant, we obtain the variational problem

$$\min_{x(x),r(x)} \int_a^b F(r, v, \lambda, \mu) dx, \quad v(0) = 0$$

with the Lagrangian

$$F = r + \lambda(m\dot{v} + \gamma v - r^q) + \mu v$$

The Euler equations are respectively (from the variation with respect to v and r)

$$\begin{aligned} \delta v : &\quad -m\dot{\lambda} + \gamma\lambda + \mu = 0 \\ \delta r : &\quad 1 + qr^{q-1}\lambda = 0 \end{aligned}$$

Solving this system, we find

$$\lambda = -\frac{\mu}{\gamma} + C\exp\left(\frac{\gamma t}{m}\right), \quad , r(t) = \frac{1}{q}\lambda^{\frac{1}{q-1}} = \frac{1}{q}\left[-\frac{\mu}{\gamma} + C\exp\left(\frac{\gamma t}{m}\right),\right]^{\frac{1}{q-1}}$$

where μ and C are still undefined constants. Those are found evaluating $v(t)$

$$v(t) = -\exp\left(-\frac{\gamma t}{m}\right) \int_0^t \frac{r(t)^q}{m} \exp\left(\frac{\gamma t}{m}\right) dt$$

and applying the integral constraint (5.47) and boundary condition $v(T) = 0$.

5.4.3 Notion of variational inequalities

The variational problem with pointwise constraints in the form of inequalities, called variational inequalities, were investigated only recently, see [**?**]. These problems are formulated as a variational problem plus an inequality.

$$\min_{u(x) \geq \phi(x)} \int_a^b F(x, u, u') dx \tag{5.48}$$

The increment of the objective functional $I(u+\delta u)-I(u)$ is nonnegative at the optimal trajectory

$$I(u+\delta u)-I(u)=-\int_a^b S(F,u)\delta u\,dx \geq 0.$$

Here, $S(F,u)$ is left-hand-term of the Euler equation (4.12). Let us analyze this formula.

When the constraint is satisfied as strict inequality, $u > \phi(x)$, an infinitesimal variation δu is not constrained and the minimizer u satisfies the Euler equation $S(F,u) = 0$ to keep the increment nonnegative. Otherwise, the extremal is coincide with the constraint, $u = \phi(x)$, variation δu of the trajectory but must be positive $\delta u \geq 0$ because all admissible trajectories $u + \delta u$ are above the constraint $\phi(x)$, $u(x) + \delta u(x) \geq \phi(x)$ for all x. Correspondingly, the variation $I(u+\delta u) - I(u)$ is nonnegative if the inequality holds

$$S(F,u)|_{u(x)=\phi(x)} \leq 0.$$

To sum up, we formulate the obtained optimality conditions. The optimal trajectory satisfies one of the two supplementary conditions:
Either

$$S(F,u)=0 \quad \text{and } u(x)\geq \phi(x)$$

or

$$u(x)=\phi(x) \quad \text{and } S(F,u)\leq 0$$

The equalities define the minimizer in each regime, and the inequalities check the optimality of the regime.

String (membrane) over an obstacle

Consider again the problem of catenoid, assuming in addition that the chain is hanged over a plane surface and is cannot go beneath it. The variational problem is

$$\min_{u(x)\geq 0, u(a)=A, u(b)=B} \int_a^b ()dx$$

and its solution is

$$\begin{array}{l} u(x)=a \quad u''\geq 0 \\ u''(x)=q \;\; u>a \end{array}$$

Convex envelope

Consider the problem about the shortest path around an obstacle discussed in Chapter **??** in geometric terms. Now we formulate the problem as a variational inequality. We find a curve $u(x)\geq 0$ that has the shortest length L

$$L=\int_a^b \sqrt{1+u'^2}dx,$$

passes through the points $(a, 0)$ and $(b, 0)$, lies over the obstacle $\phi(x)$

$$u(x) \geq \phi(x), \quad \forall x \in [a, b]$$

Remark 5.4.2 We assume that the equation of the obstacle $\phi(x)$ is defined for all $x \in [a, b]$. If it is not defined for some x, we put $\phi(x) = 0$ for these x.

The Euler equation $S(F, u) = 0$ corresponds to the operator

$$S(F, u) = \frac{d}{dx} \frac{u'}{\sqrt{1 + u'^2}} = \frac{u''}{\sqrt{1 + u'^2}^3}$$

its sign coincide with the sign of u'',

$$S(F, u) = A^2 u'', \quad \text{where } A = \frac{1}{(1 + u'^2)^{\frac{3}{4}}} > 0$$

The extremal is found from the conditions (**??**) which take the form:

$$\begin{array}{llll} \text{Either} & u(x) = \phi(x) & \text{and} & u'' \leq 0, \\ \text{or} & u'' = 0 & \text{and} & u(x) > \phi(x) \end{array}$$

Multidimensional version The problem of the convex envelope of a function of a vector argument can be formulated as the variational inequality as well. The conditions of convexity of a differentiable function are

$$\begin{array}{ll} u(x) = f(x) \quad H(u) \geq 0 & \\ \det H(u) = 0, \ H(u) \geq 0, & u(x) < f(x) \end{array}$$

This problem will be discussed in Chapter **??**

5.4.4 Summary

1. Euler equations and Lagrange method in variational problems can be viewed as limits of stationary conditions of a finite-dimensional minimization problem.

2. Lagrange method allows to solve isoperimetric or constrained extremal problem of a rather general form, reducing it to the canonic variational problem. The solution is first defined as a function of the unknown multipliers, which are later determined from the constraints. Alternatively, the multipliers can be algebraically excluded from the optimality condition.

3. The total number of boundary conditions in a variational problem always matches the order of differential equations. The boundary conditions are either initially given or are obtained from the minimization requirement.

4. The length of the interval of integration, if unknown, also can be obtained from the minimization requirement.

We will observe these features also in the optimization of multiple integrals: The variational problems supply of missing components of the problem formulation. We will see that they also can make the solution "better" that is more stable and even can help define the solution to the problem where no solution exist.