Domino Magnification

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Abstract

The conditions are investigated under which a row of increasing dominoes is able to keep tumbling over. The analysis is restricted to the simplest case of frictionless dominoes that only can topple not slide. The model is scale invariant, i.e. dominoes and distances grow in size at a fixed rate, while keeping the aspect ratios of the dominoes constant. The maximal magnification factor for which a domino effect exist is determined as a function of the mutual separation.

1 Introduction

The domino effect has received considerable attention because it is spectacular and seems to be governed by simple physics [1, 2, 3]. However, careful experiments [4, 5, 6] are not so easy. Measuring the propagation speed of the order of a meter per second of a train of falling dominoes, is already too difficult to observe by elementary means. Equally, the theory of toppling dominoes is not as simple as the phenomenon looks like and cannot be reduced to the straightforward application of the conservation laws of energy and angular momentum.

The discussion of a row of falling dominoes involves two aspects: the tumbling motion and the collisions. The first to realize that the domino effect is a collective phenomenon was D. E. Shaw [7]. He noted that the dominoes lean on each other after collision. Consequently the collision of dominoes is fully inelastic and this is the main source of energy losses during the process. It is tempting to invoke conservation of angular momentum at the collision as in [7], but this does not apply, since colliding dominoes topple around different axes [8, 9].

The reason to come back on the domino effect is a question posed at the Dutch Science Quiz 2012 reading: "*how many dominoes does one need to topple a domino as tall as the Domtoren?*" (a tower of 112 meter high). A restriction was that the dominoes have the same aspect ratios as that of standard dominoes. This restriction is essential for an unique answer, since the thinner the domino (with fixed height) the easier it topples. The idea behind the question is that every successive domino is a factor r larger than the preceding one. The dominoes grow in size as a geometric series r^n and the correct answer is the n that makes r^n times the height of a standard domino equal to 112 meter. Whitehead [11] describes a demonstration with a magnification factor $r = 1.5$.

In order to turn this problem into a scientific problem, one has to make further specifications. One is a rule for the distance between successive dominoes. Although not specified in the problem, the logical choice is to scale the distance with the size. To let the distance grow with the size of the domino is reasonable, the larger the domino, the more space one needs between them. Moreover, there will be an optimal distance in each cycle and this optimum will scale with the size of the dominoes. So we restrict ourselves to a scale invariant model.

In order to make the mechanical model precise, a few more conditions have to be imposed:

1. The dominoes may not slide with respect to the ground. It is clear that sliding is an energy loss that impedes the domino effect. Infinite friction with the floor is favorable and can basically be reached in practice. Thus the only possible motion is to topple over. So the state of domino is given by a single angle θ measuring the deviation from the normal to the ground.

2. The dominoes stay in contact with each other after the collision [7]. This makes the collisions fully inelastic. In practice this condition is fulfilled as the movie of the the demonstration with wooden dominoes shows [12]. Inelastic collisions are the main energy drain. Steel dominoes would do better, but are too costly to produce in large sizes. We take this condition as a constraint on the maximum possible r.

3. After hitting a new domino, the dominoes slide frictionless over each other. It is of course not realized with wooden dominoes, but it would be not difficult to smooth the surfaces and minimize the friction by lubrication. Inclusion of friction is quite well possible [9, 10], leading to a substantial complication in the calculation, while adding little to the understanding of the mechanism.

We start the discussion with the elementary process of one domino colliding with another at distance s and tumbling together downwards. This provides the setting for the longer train of falling dominoes. The dimensions of the dominoes are denoted by height h, the thickness d and the width w. We will consistently work with dimensionless distances to ease the calculations. The parameter w/h hardly enters in the calculation. The width is only co-determinant for the mass m_i of the dominoes. As in all gravitational phenomena, the total mass of the object drops out of the equations, but not the mass distribution. We consider two (extreme) cases: $q = 4$, referring to massive dominoes where the mass is proportional to the volume and $q = 3$, referring to hollow dominoes, with a mass proportional to the surface. The latter case seems curious, because standard dominoes are massive. However in an attempt to topple a very large domino [12], the larger dominoes have to be hollow for practical reasons.

As mentioned the dominoes and their mutual distance all have the same aspect ratios: $w/h, s/h$ and $d/h = 0.14583$, which is taken equal to the ratio of standard dominoes. Each new domino is r times larger than its predecessor. One could consider varying s and r; it would make the discussion considerably more involved and not so much richer, since we investigate the largest possible magnification factor and it turns out that this occurs at a well defined value of s. So deviating from the optimal s and r makes the domino effect less effective. Restricting ourselves to the scale invariant case allows to discuss only one cycle in the domino effect, all the others are similar.

A dimensionless time is somewhat more delicate. The natural time unit reads

$$\tau = \sqrt{I/mgh} = \sqrt{(h^2 + d^2)/(3gh)}, \tag{1}$$

where I is the moment of inertia of the foremost domino. This is the appropriate time scale for one cycle, the next is a factor $\sqrt{r}$ larger, the previous a factor $\sqrt{r}$ smaller. In the formulas we will not express time in terms of the unit τ, as this complicates the connection between different cycles.

The description of the train of falling dominoes centers around the tilt angle θ of the foremost domino as function of the time t. It starts at $\theta(0) = 0$ and ends at t_f at collision angle $\theta(t_f) = \theta_c$, where the foremost domino looses its role as head of the falling train of dominoes. θ_c is given by

$$\sin\theta_c = \frac{r\,s}{h}. \tag{2}$$

After θ_c, the domino continues to fall, but it becomes a slave, just as the previous domino was of the foremost. We number the dominoes with respect to the foremost falling domino, which gets the number 0, or by default no number. So $\theta = \theta_0$ is the tilt angle of the foremost domino, θ_1 that of its predecessor and so on. The calculation of t_f gives the propagation speed of the domino effect. It is of no concern here. We only note that t_f will be proportional to τ defined in (1) and therefore t_f will increase with a factor $\sqrt{r}$ in every cycle. This slowing down of each cycle is the beauty of the record attempt [12], where one can see *ad oculos* the details of a cycle.

The derivative of θ with respect to t is the angular velocity

$$\omega = \frac{d\theta}{dt}. \tag{3}$$

As long as ω is positive the domino train proceeds. So the question is whether ω remains positive till the foremost collides with the next one. In order to calculate ω we have to understand what happens during a collision and what is the equation of motion between collisions. The central function in this calculation is the relation between the angle θ_1 of domino 1 as function of the angle θ of the leading domino 0.

We start the discussion with two dominoes 0 and 1. To ease the notation we replace from now on s/h by s and d/h by d.

2 Two dominoes

Consider domino 1, freely rotating towards the still upright domino 0. In order that domino 1 keeps moving, the initial push must be large enough

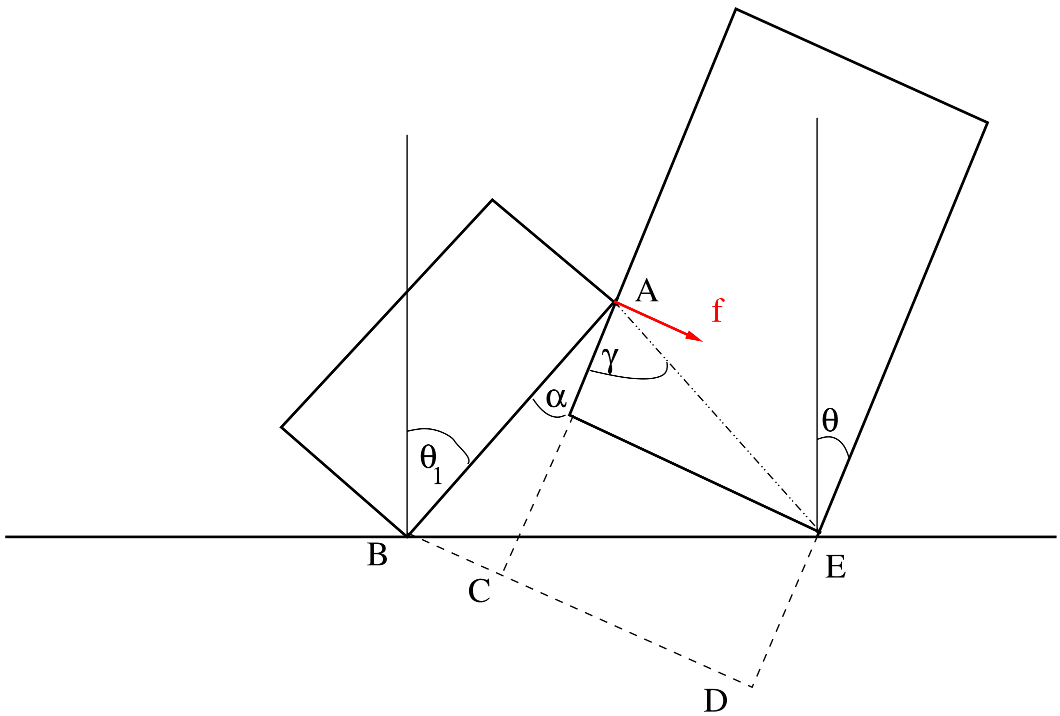

Figure 1: Successive dominoes. The tilt angle θ is taken with respect to the vertical. Domino 1 hits 0 at the point A. The rotation axis of 1 is the point B and E is that of 0. The normal force f that domino 1 exerts on domino 0 is also indicated. The goniometric relations are summarized in Table 1.

to overcome the point of highest potential energy, occuring at the angle θ_u, given by

$$\sin\theta_u = \frac{d}{\sqrt{1+d^2}}. \tag{4}$$

Here we assume that domino 1 reaches this point before that it hits domino 0. So we must have $\theta_u < \theta_c$. With (2) and (4) this implies

$$s > \frac{d}{r\sqrt{1+d^2}}. \tag{5}$$

Provided that s fulfils this criterion, we find for the maximum potential energy

$$E_{1u} = \frac{1}{2} m_1 g h_1 \left(\cos\theta_{1u} + d\sin\theta_{1u}\right). \tag{6}$$

Under marginal circumstances, domino 1 has a vanishing rotation at this point and thus is its total energy also E_{1u}. While falling the potential energy decreases and reaches at the collision angle θ_c the value

$$E_{1c} = \frac{1}{2} m_1 g h_1 (\cos\theta_c + d\sin\theta_c). \tag{7}$$

The energy difference is converted into kinetic energy. At the collision the kinetic energy amounts

$$\frac{1}{2} I_1\, \omega_{1c}^2 = E_{1u} - E_{1c}, \tag{8}$$

where I_1 is the moment of inertia of domino 1. So the angular velocity at the collision is given by (with τ_1 the time scale (1) for domino 1)

$$\tau_1^2 \omega_{1c}^2 = (\cos\theta_{1u} + d\sin\theta_{1u}) - (\cos\theta_c + d\sin\theta_c). \tag{9}$$

We consider the collision as instantaneous. Domino 0 gets an impuls F from 1, changing the angular velocity of domino 0 abruptly from 0 to ω_0, given by

$$I\omega_0 = Fa, \tag{10}$$

where a is the arm of the torque exerted on domino 0. Through the principle action is reaction, domino 0 gives an impuls $-F$ to domino 1, such that its angular velocity changes from ω_{1c} instantaneously to ω_0 (as we will see in a moment). So we have the equation

$$I_1(\omega_0 - \omega_{1c}) = -Fb, \tag{11}$$

where b is the moment arm of the torque on domino 1 with respect to its turning point. For the computation of a and b we have to rely on the goniometry between touching dominoes which is sketched in Fig. 1.

quantity	formula
top angle α of rectangular triangle ABC	$\alpha = \theta_1 - \theta$
base BC of triangle ABC	$\mathrm{BC} = h\sin\alpha$
top angle β of rectangular triangle EBD	$\beta = \pi/2 - \theta$
base BD of triangle EBD	$\mathrm{BD} = (s+d)\sin\beta$
height h_{cm} of center of mass domino 0	$2h_{cm} = h\cos\theta + d\sin\theta$
moment arm b of force $-f$ exerted on 1	$b = (h/r)\cos\alpha$
moment arm a of force f exerted	$a = \mathrm{AE}\sin\gamma = \mathrm{AC\text{-}DE}$
domino 0	$= (h/r)\cos\alpha - (s+d)\sin\theta$

Table 1: Goniometric relations referring to Fig. 1

The various goniometric relations are explained in Table 1. The basic relation between the angle θ_1 and θ follows by equating the given expression for BC and for BC as BD$-d$

$$(1/r)\sin(\theta_1 - \theta) = (s+d)\cos\theta - d. \tag{12}$$

It is a direct consequence of the condition that the dominoes lean on each other after a collision. (12) defines θ_1 as a function $\theta_1(\theta)$ of θ in the interval $0 \le \theta \le \theta_c$.

The moment arms a and b, as defined in Table 1, are obviously equal at collision where $\theta = 0$. The angular velocity of domino 1 just after collision, equals that of domino 0. To see this we have to determine the relation between ω_1 and ω, when the motion of domino 1 is a slave of domino 0. Generally holds

$$\omega_1 = \frac{d\theta_1}{dt} = \frac{d\theta_1}{d\theta}\,\omega = \theta_1'(\theta)\,\omega. \qquad (13)$$

The prime denotes differentiation with respect to the argument and is used for shortness. Differentiation of (12) with respect to θ gives

$$\theta_1'(\theta) = \frac{(1/r)\cos(\theta_1 - \theta) - (s + d)\sin\theta}{(1/r)\cos(\theta_1 - \theta)} = \frac{a}{b}. \qquad (14)$$

The last equality is demonstrated in Table 1. Using this relation for $\theta = 0$, where $a = b$, leads to $\theta_1'(0) = 1$, showing that the two angular velocities are equal at collision.

We use arguments for the angle θ and subscripts for corresponding time. So ω_{1c}, in (9), is the angular velocity of domino 1 just before the collision with 0 and $\omega_1(0)$ is its value just after the collision has taken place.

The equality $a = b$ (at collision) makes it easy to eliminate the impuls F, yielding the following relation between ω_0 and ω_{1c}

$$(I + I_1)\,\omega_0 = I_1\,\omega_{1c}. \qquad (15)$$

Here we see that the ratio of the moments of inertia of the two dominoes is the important ingredient for the collision. It equals r^{-q-1} with $q = 4$ for massive dominoes and $q = 3$ for hollow dominoes.

Now ω_0 must be large enough to get the combination of the two sliding dominoes over the maximum of their combined potential energy, which equals

$$V(\theta) = \frac{1}{2}mgh\,[r^{-q}(cos\theta_1 + d\sin\theta_1) + \cos\theta + d\sin\theta]. \qquad (16)$$

Note that domino 1 weighs by a factor r^{-q} less in the sum. We get a condition for the domino effect by requiring that the kinetic energy of the pair 1 and 0 must be larger than the potential barrier. With $\omega_1(0) = \omega_0$ the condition reads

$$\frac{1}{2}(I + I_1)\,\omega_0^2 \geq [V(\theta_m) - V(0)], \qquad (17)$$

where θ_m is the tilt angle where the maximum of $P(\theta)$ is reached. It will be close the angle θ_u where domino 0 reaches its maximum potential energy, but the contribution of domino 1 is decreasing at this point and so θ_m is (slightly) smaller than θ_u (as the contribution of domino 1 only counts by a factor r^{-q} less).

We now have sufficient equations to determine maximum factor r still allowing the domino 1 to topple over domino 2: the expression (9) for ω_{1c}, the relation (15) between ω_0 and ω_{1c} and the condition (17) for ω_0. τ_1 is related to τ by

$$\tau_1^2 = \tau^2/r. \qquad (18)$$

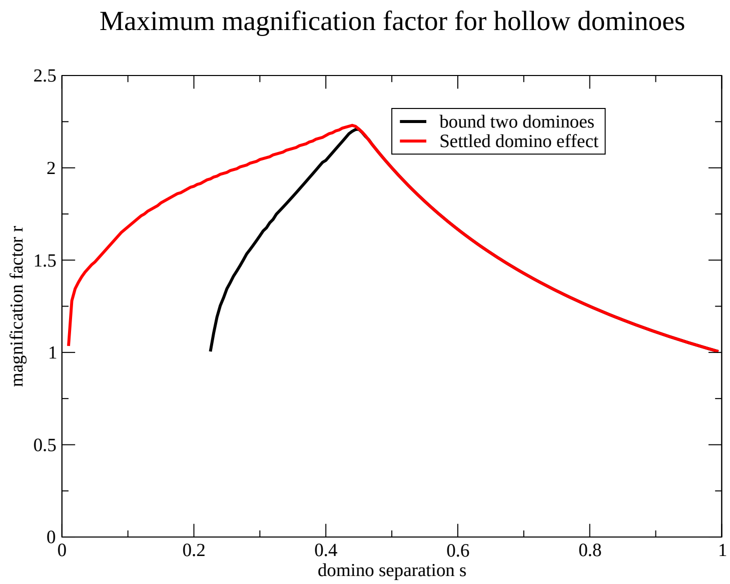

Figure 2: Bounds on the marginal domino effect for hollow dominoes.

The above equations impose on the parameter r a condition for given s. The boundary, for which the domino effect becomes marginal, is given by the equality in (17). Although the equations can written out explicitly, their solution is too complicated to discuss analytically, because the angle θ_m cannot be found analytically. However it is simple to do it numerically. Start with a value of the separation s (well beyond the restriction (5)) and a low value for r for which we are sure that the condition (17) is fulfilled. Then determine θ_m by slowly raising θ from $\theta = 0$ and see whether the maximum $V(\theta)$ is reached. Once the $V(\theta)$ starts to decrease, one is beyond the maximum. It may happen right away at $\theta = 0$ and then the maximum is $\theta_m = 0$, where (17) is surely fulfilled. Otherwise check whether (17) is fulfilled and if so raise r. If not, one is beyond the boundary, which then can be determined more accurately by interpolating beteen the present r and the previous r. The outcome of this exercise is shown for hollow dominoes in Fig. 2. The flank of the curve at the large-separation side is trivial: the domino must be large enough to hit the next domino.

3 A general domino train

The discussion of the previous section suffers from the fact that we have assumed that the initial push to domino 1 is marginal, i.e. just sufficient to get it over its own potential barrier. The bound following from this condition is too restrictive in practice. We now assume that the initial push is large enough to set the train into motion and ask the question whether the train can sustain itself, i.e. whether the total kinetic energy after collision with a new domino, is large enough to get the enlarged collection over its potential barrier. In every cycle the potential barrier can only shift to smaller angles θ, since at the tail, one more domino is in the train, which is already (long) over its potential maximum. Once the maximum occurs at $\theta = 0$, there is no question anymore whether the new train will make it, since it runs only downhill.

The analysis of this problems runs to a large extent along the same lines as in the previous section. We divide out the mass and are concerned about the total height contribution to the potential energy of a train of N dominoes leaning on each other. It follows as the sum

$$H(\theta) = \sum_{i=0}^{N-1} r^{-qi} \left[\cos\theta_i(\theta) + d\sin\theta_i(\theta)\right]. \tag{19}$$

Likewise the total kinetic energy can be represented by

$$J(\theta) = \sum_{i=0}^{N-1} r^{-(q+1)i} \left[\theta_i'(\theta)\right]^2. \tag{20}$$

Here $\theta_i(\theta)$ is the tilt angle of domino i as function of the tilt angle θ of the foremost and $\theta_i'(\theta)$ is its derivative. In appendix B we derive that conservation of energy between collisions implies relation (51) or

$$J(\theta)\,\tau^2\omega^2(\theta) = H(0) - H(\theta) + J(0)\,\tau^2\omega^2(0). \tag{21}$$

This enables to calculate $\omega(\theta)$ for given $\omega(0)$.

(21) also yields the initial value $\omega(0)$ if we combine it with the collision equation and the scale invariance. In appendix C we show that the angular velocity ω_0 of the foremost domino, just after being hit, is related to the angular velocity ω_{1c} of the hitting domino as

$$J(0)\,\omega(0) = (J(0) - 1)\,\omega_1(\theta_c). \tag{22}$$

So far this is just a generalization from 2 dominoes to N dominoes in the train. The new element is that we require that the cycles are self-similar.

$$\tau_1\,\omega_1(\theta) = \tau\,\omega(\theta) \qquad \text{or} \qquad \omega_1(\theta) = \sqrt{r}\,\omega(\theta) \tag{23}$$

The three equations (21-23) determine the values of $\omega(0)$.

$$\tau^2\,\omega^2(0) = \frac{P\,(J(0) - 1)}{J(0)[r^q J(0) - J(0) + 1]}, \tag{24}$$

Here P is the "fuel" of the domino effect: the difference between the potential energy of an upright domino and a fallen domino.

$$P = P(h, d, s) = H(0) - H(\theta_c) = 1 - \cos\theta_f - d\sin\theta_f = 1 - x_f - dy_f. \tag{25}$$

The value of θ_f is given in (42) of Appendix A. In (24) we have eliminated the value $J(\theta_c)$ using relation (50).

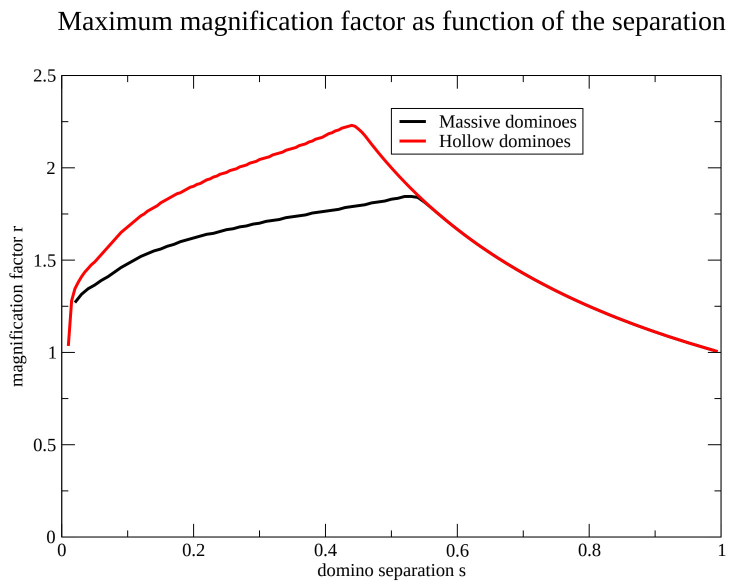

Figure 3: Magnification factor as function of the separation.

With these equations one can check whether the kinetic energy of the train after collision, exceeds the potential increase from $\theta = 0$ to its maximum at θ_m

$$\tau^2 \omega_0^2 \geq [H(\theta_m) - H(0)], \tag{26}$$

We now summarize the steps to find the maximum r with a domino effect as function of s. In the Appendices the derivations and additional expressions are given.

1. Compute for a given s and r the initial value of x_f as given by (42) or (40), given in Appendix A, depending on the condition (41).

2. Start the iteration scheme for the $\theta_i(\theta)$ and $\theta_i'(\theta)$, as described in Appendix A, for the angle $\theta = 0$. This yields the value of $J(0)$. Compute with (25) the value of the fuel P and with (24) the initial angular velocity of a cycle $\omega(0)$.

3. Then search with equation (19) for the value θ_m for which $H(\theta)$ is maximal and find the corresponding minimal $\omega(\theta_m)$ with (21).

4. If $\omega(\theta_m)$ is still positive, raise the value of r and repeat the previous steps 2-4 in order to find again the minimal angular velocity $\omega(\theta_m)$. Keep raising r till the minimum becomes zero. This gives the maximum magnification factor for the chosen value of s

5. Do this story for $0 < s < 1$

The result of the calculation is shown in Fig. 3 for hollow and massive dominoes. The curve consists of a rising part for $s < 0.5$ an a drop-off as $r = 1/s$ for higher values. The latter part of the curve reflects the condition the foremost domino must be hit by the previous in order to be toppled. This limit is fairly unrealistic, but follows from our condition that the dominoes cannot slip over the ground but only topple. Near that line, the foremost domino hits the next at a low point, such that the tendency to slip is larger than to topple, as the moment arm becomes very small.

A Relations between the falling dominoes

This Appendix discusses the geometric relations between the row of falling dominoes that lean on each other. In Section 2 we gave in equation (12) the relation between the tilt angle θ_1 of domino 1 and the θ of domino 0. The successively higher numbered dominoes follow from the same relation as domino i is related in exactly the same way to $i-1$ as 1 to 0. In general

$$\theta_i(\theta) = \theta_1(\theta_{i-1}) = \theta_1(\theta_1(\theta_{i-2}) = \theta_1(\theta_1(\theta_1(\cdots(\theta)))) \tag{27}$$

Thus equation (27) gives all the tilt angles of the followers in terms of θ. Rather then using the explicit relation

$$\theta_i(\theta) = \theta_{i-1}(\theta) + \arcsin((r/h)[(s+d)\cos\theta_{i-1}(\theta) - d]), \tag{28}$$

we construct the relation in terms of a rotation in carthesian coordinates

$$x_i = \sin\theta_i, \quad y_i = \cos\theta_i. \tag{29}$$

With the definitions

$$X_i = \sin(\theta_{i+1} - \theta_i) = (r/h)[(s+d)\,y_i - d], \quad Y_i = \cos(\theta_{i+1} - \theta_i) = \sqrt{1 - X_i^2}, \tag{30}$$

the rotation equations read (using $\theta_{i+1} = \theta_i + (\theta_{i+1} - \theta_i)$)

$$x_{i+1} = Y_i\,x_i + X_i\,y_i, \qquad y_{i+1} = -X_i\,x_i + Y_i\,x_i. \tag{31}$$

These equations are the same for every pair of successive dominoes. Note that this scheme contains only multiplications and one square root and not any goniometric function, which speeds up the iteration.

The function $\theta_1(\theta)$ is the central ingredient for the calculations. So we list a few more of its properties. As $\theta = 0$ corresponds to the collision angle θ_c for domino 1 we have

$$\theta_1(0) = \theta_c. \tag{32}$$

By differentiating (27) with respect to θ, we construct a recursion relation between the derivatives

$$\theta'_{i+1}(\theta) = \frac{d\theta_1(\theta_i)}{d\theta_i}\,\theta'_i(\theta). \tag{33}$$

For the first factor we can use (14) in the form

$$\frac{d\theta_1(\theta_i)}{d\theta_i} = \frac{(1/r)\,Y_i - (s+d)\,x_i}{(1/r)Y_i} = \frac{a_i}{b_i}, \tag{34}$$

with the convention the indices 0 apply to the foremost. We cast the combination of (34) and (33) in the convenient recursive form

$$\theta'_{i+1}\, b_i = \theta'_i\, a_i. \tag{35}$$

As last useful formula we write (27) as

$$\theta_i(\theta) = \theta_{i-1}(\theta_1(\theta)) \tag{36}$$

and differentiate with respect to θ, yielding for the derivative

$$\theta'_i(\theta) = \theta'_{i-1}(\theta_1(\theta))\, \theta'_1(\theta). \tag{37}$$

We will use this expression for $\theta = 0$ with the result, using (32) and (14)

$$\theta'_i(0) = \theta'_{i-1}(\theta_c). \tag{38}$$

The derivatives of the θ_i are employed in the expression of the angular velocities ω_i in terms of ω

$$\omega_i = \frac{d\theta_i}{d\theta}\frac{d\theta}{dt} = \theta'_i\, \omega. \tag{39}$$

This completes the discussion of the properties of the falling dominoes in terms of the parameters θ and ω of the foremost. The transformations (31) generate for a given θ the sequence of θ_i or rather the sets x_i, y_i and X_i, Y_i and with these values we can iterate simultaneously the angular velocities ω_i in terms of ω. In principle we would have to iterate ∞ many steps, but the tilt angles rapidly approach the stacking angle θ_∞, following as

$$y_\infty = \cos\theta_\infty = d/(s+d). \tag{40}$$

The stacking angle is the angle in which a static row of dominoes leans on each other for a given separation s. So a limited number of iteration steps suffices. In fact for rather wide separation the stacking angle is not realized, because the previous domino is too small to lean on the next. This happens when

$$(1/r)^2 < (s+d)^2 - d^2. \tag{41}$$

The last tilt angle θ_f which enables leaning is given by

$$x_f = \sin\theta_f = \frac{(1/r)^2 + (s+d)^2 - d^2}{2(1/r)(s+d)}. \tag{42}$$

As soon as x_i exceeds x_f, one must end the iteration, because for larger angles the domino does not lean anymore, but falls free and does not contribute to the mechanics of the domino effect. In case condition (41) is fulfilled, we take $x_\infty = x_f$.

B The motion between collisions

The domino effect is a succession between tumbling and colliding. In this Appendix we consider the phase of tumbling. As the dominoes slide frictionless over each other, the motion is completely determined by the conservation of energy. Energy is lost in the collissions. The energy has a kinetic and a potential part. The potential energy of domino i is given by

$$V_i = \frac{1}{2} mg\,(h\,y_i + d\,x_i)\,r^{-qi} \tag{43}$$

The power of r^{-qi} comes in the expression because the mass goes down with a factor r^{1-q} and the size with r^{-1} in every step backwards down the train. The kinetic energy of domino I reads

$$K_i = \frac{1}{2} I_i \omega_i^2 \tag{44}$$

with the moment of inertia I_i

$$I_i = m_i\,[h_i^2 + d_i^2]/3 = m\,[h^2 + d^2]\,r^{-(q+1)i}/3 = I\,r^{-(q+1)i}. \tag{45}$$

The $q + 1$ power of the magnification factor follows from the mass and the size squared. The total energy of the falling train then equals

$$E = \sum_i (K_i + V_i). \tag{46}$$

We make the energy dimensionless by dividing it by $mgh/2$ yielding

$$\epsilon = \frac{2E}{mgh} = H(\theta) + \tau^2\,J(\theta)\,\omega^2, \tag{47}$$

with the total dimensionless potential energy H

$$H(\theta) = \sum_i [x_i + (d/h)y_i]\,r^{-qi} \equiv H(\theta) \tag{48}$$

and the total dimensionless moment of inertia is

$$J(\theta) = \sum_i r^{-(q+1)i}\,[\theta_i'(\theta)]^2 \equiv J(\theta). \tag{49}$$

For a given value of θ all the ingredients for the functions $H(\theta)$ and $J(\theta)$ are well defined by the recursion relations of the previous section. We need in Section 3 the values of $J(\theta)$ for the angles $\theta = 0$ and $\theta = \theta_c$. Using (38) one finds the relation

$$J(0) = 1 + r^{-(q+1)} \qquad \text{or} \qquad J(\theta_c) = r^{q+1}[J(0) - 1]. \tag{50}$$

Conservation of energy implies that ϵ is independent of time of the tilt angle, such that we have

$$\epsilon = H(0) + J(0)\tau^2\omega^2(0) = H(\theta) + J(\theta)\tau^2\omega^2(\theta).. \tag{51}$$

This allows to find $\omega(\theta)$ for a given $\omega(0)$.

C The collision equation

The stages of rotational motion are connected by collisions. In this Appendix we compute the initial value $\omega(0)$ from the final value $\omega_(\theta_c)$ of the previous cycle. As we mentioned in the introduction, we assume that the two colliding dominoes stick together after the collision. That means that in the in the center of mass system the relative kinetic energy is completed dissipated. In other words the collision is fully inelastic. We assume that we have a fully developed domino train such that the cycles are the similar (upon a factor $\sqrt{r}$): before the collision the hitting domino has an angular velocity $\omega_1(\theta_c) = \sqrt{r}\omega(\theta_c)$ and the upright domino gets a velocity $\omega(0)$ after the collision.

The idea is that during the collision, forces are exerted in a very short time span, such that the angles do not change during the collision. Instead the angular velocities make a jump. When domino 1 hits 0, its own angular velocity is suddenly reduced and that of 0 jumps to the non-zero value $\omega(0)$. The jumps in the angular velocity propagate downwards in magnitude, in order to keep the dominoes in contact. Therefore the impulses have to propagate downwards in order to realize these jumps. The jump of the foremost domino is from $\omega = 0$ before the collision to $\omega(0)$ after the collision

$$I\Delta\omega_0 = I\omega(0) = F_0 a_0 \tag{52}$$

The second equality gives the integrated torque during the collision, which is the impuls F_0 exerted by domino 1 on domino 0 times the arm a_0 with respect to the rotation axis of 0. Likewise the domino 1 feels the impuls $-F_0$ from domino 0 and F_1 from domino 2.

$$I_1\Delta\omega_1 = I_1\left[\theta_1'(0)\,\omega(0) - \theta_0'(\theta_c)\,\omega_1(\theta_c)\right] = F_1 a_1 - F_0 b_0. \tag{53}$$

The first term gives the value of ω_1 as calculated from the foremost domino 0. The second term is the angular velocity of 1 before the collision when domino 1 was the foremost with angle θ_c. The factor $\theta_0'(\theta_c) = 1$ is there for reason of generality and the index 1 in $\omega_1(\theta_c)$ is to indicate that here domino 1 is seen as the foremost. In general domino i receives $-F_{i-1}$ from $i-1$ and F_i from $i+1$. As general equation we get

$$I_i\Delta\omega_i = I_i\left[\theta_i'(0)\,\omega(0) - \ \theta_{i-1}'(\theta_c)\,\omega_1(\theta_c)\right] = F_i a_i - F_{i-1} b_{i-1}. \tag{54}$$

We see that impuls F_i occurs in two equations: for domino $i+1$ with a minus sign and for domino i with a plus sign. We can eliminate the impulses by multiplying equation for domino i with the factor s_i and add them up. If s_i has the property

$$s_i a_i = s_{i+1} b_i \tag{55}$$

the impulses drop out of the sum. We take $s_0 = 1$. Comparing this recursion for the s_i with the recursion (35) for the derivatives of the tilt angles we may identify

$$s_i = \theta_i'(0). \tag{56}$$

The argument $\theta = 0$ occurs in this formula, because the moment arms are taken in the situation where the foremost domino is hit and thus still has a tilt angle $\theta = 0$.

Using (56) the coefficient of $\omega(0)$ becomes

$$\sum_{i=0} I_i \, [\theta_i'(0)]^2 = I \, J(0). \tag{57}$$

The coefficient of $\omega_1(\theta_c)$ reads with (38)

$$\sum_{i=1} \theta_i'(0) I_i \, \theta_{i-1}'(\theta_c) = \sum_{i=1} I_i \, [\theta_i'(0)]^2 = I \, (J(0) - 1), \tag{58}$$

such that we obtain the collision law

$$J(0) \, \omega(0) = (J(0) - 1) \, \omega_1(\theta_c). \tag{59}$$

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