

# Domino Waves\*

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**Abstract.** Motivated by a proposal of Daykin [Problem 71-19\*, *SIAM Rev.*, 13 (1971), pp. 569–570], we study the wave that propagates along an infinite chain of dominoes and find the limiting speed of the wave in an extreme case.

**Key words.** dominoes, waves, modeling, mechanics

**AMS subject classifications.** 70B99, 70F35, 97A90

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**I. Introduction.** Everyone is familiar with dominoes and has used them for fun. A common game is to arrange the dominoes in a row (Figure 1.1) and give a push to the first. This generates a pleasing wave of falling dominoes. The propagation happens at some speed  $v$  (not necessarily constant). A qualitative discussion for general audiences is given by Walker in [3]. Given the game’s simplicity it is perhaps surprising to discover that an exact computation of the speed  $v$  is quite difficult. Daykin realized this in the following 1971 proposal ([1]; see also [2]) to the readers of the *SIAM Review*:

“How fast do dominoes fall?”

The “domino theory of Southeast Asia” says that if Vietnam falls, then Laos falls, then Cambodia falls, and so on. Hearing a discussion of the theory led me to wonder about the proposed physical problem. The reader is invited to set his or her own “reasonable” simplifying assumptions, such as perfectly elastic dominoes, constant coefficient of friction between dominoes and the table, and initial configuration with the dominoes equally spaced in a straight line, and so on.

In 1983, McLachlan et al. [4] found a scaling law for the speed  $v$  in the limiting case of dominoes with zero thickness equally spaced in a straight line. With these assumptions the authors found the functional relation

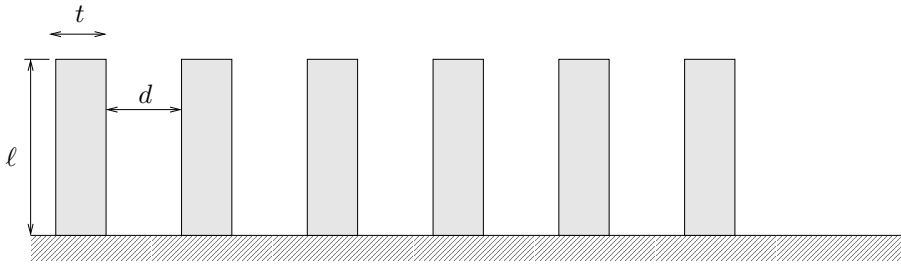
$$(1.1) \quad v = \sqrt{g\ell} G\left(\frac{d}{\ell}\right).$$

Here  $\ell$  is the height of the dominoes,  $d$  the spacing between dominoes, and  $G(x)$  an undetermined function of  $x$ . This relation followed from dimensional analysis of

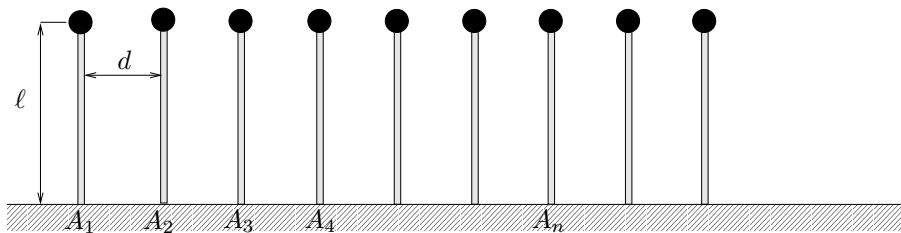
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**Fig. 1.1** A uniform chain of dominoes.



**Fig. 1.2** A chain of massless rods carrying masses  $m$  on top. The masses are indicated as finite spheres only for the sake of visualization.

the problem.<sup>1</sup> McLachlan et al. proceeded to test the formula experimentally using dominoes of heights  $h = 4.445$  cm and  $h = 8.890$  cm. More recently, Banks presented a simplified description of the effect [5]. His analysis, among other assumptions, assumes a uniform propagation speed and conservation of momentum. A different direction was taken by Shaw in a short paper [6] describing how to model the domino effect as a computer simulation and use it as an experiment in an undergraduate physics lab.

Equation (1.1) is not an entirely satisfactory solution, of course, containing as it does an unknown function. This recognition became the motivation for the present article. In this work we develop an expression for the speed  $v$ . The result can be cast as a particular form of the scaling function  $G$  arising from a particular set of assumptions.

In order to highlight the basic physics behind the problem, we replace the dominoes by massless rods topped with point masses  $m$ , as seen in Figure 1.2. A similar analysis in the case of dominoes with the shape of parallelepipeds is straightforward, although there are some minor differences.

## 2. The Model.

### 2.1. The Assumptions.

- We shall assume the following:
1. *The chain of rods is uniform.* This means that all rods are identical and are equally spaced along a straight line. Let  $\ell$  be the length of a rod and  $m$  the mass on top.
  2. *The collisions are head-on.* This means that, seen from above, all the rods are and remain aligned on the same line. If this is not the case, then additional

<sup>1</sup>This paper should allow the reader to write down a complete list of assumptions needed to reach this conclusion.

parameters are necessary to describe the collisions from point to point. The higher the asymmetry, the more parameters are needed, and the problem becomes highly complicated.

3. There is *enough static friction* between the rods and the floor to keep the rods from sliding relative to the floor. Thus the rods pivot about fixed axes.
4. *No energy is dissipated at the contact point* between the rods and the floor. This condition is independent of the previous one; it is possible for an object to rotate about an axis and yet dissipate energy.
5. *Collisions are instantaneous*. This means that the time interval  $\Delta t$  during which a collision occurs is zero. The change in total angular momentum is then  $\Delta \vec{L} = \int_0^{\Delta t} \vec{\tau} dt = 0$ . That is, total angular momentum is conserved during the collision, even though gravity is an external force and produces a nonzero torque.
6. *The collisions are elastic*. This means that energy is not dissipated during the collisions.
7. *The rods are stiff*. This means that there is no deformation of the rods and thus no energy is converted to elastic potential energy of deformation. This condition is independent of the previous one; it is possible for the rods to be stiff and still dissipate energy during a collision.

**2.2. Definition of Symbols.** To facilitate the calculations in the next section, we present our notation in advance. We label the rods sequentially with the numbers  $1, 2, 3, \dots, k, \dots$ , starting with 1 from the left end. We also label  $A_1, A_2, A_3, \dots, A_k, \dots$ , the pivot points of the rods as seen in Figure 1.2. Then

- $\theta_k$  is the angular displacement of a rod from the vertical, and  $\omega_k = d\theta_k/dt$  is the corresponding time-dependent angular velocity;
- $\Omega_1$  is the initial angular velocity of the first rod immediately after it is pushed;
- $\Omega_k$  is the initial angular velocity of the  $k$ th rod just as it begins to move; this is of course the result of the collision with the  $(k-1)$ th rod (for  $k > 1$ );
- $\Omega_{fk}$  is the angular velocity of the  $k$ th rod just before collision with the  $(k+1)$ th rod (the subscript  $f$  means “final” or “fallen”);
- $\Omega_{bk}$  is the angular velocity of the  $k$ th rod just after collision with the  $(k+1)$ th rod ( $b$  because the rod has just “bounced”);
- $\beta_1$  is the angle a rod forms with the vertical at the point of collision:

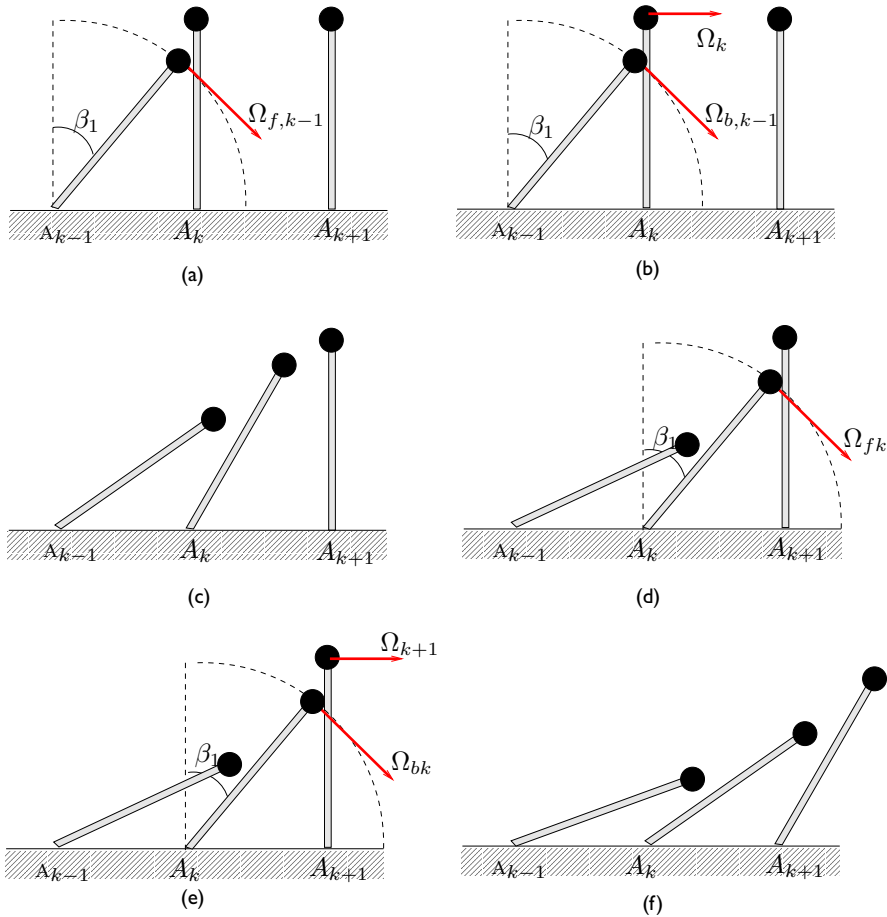
$$\beta_1 = \sin^{-1} \frac{d}{\ell};$$

- $T_k$  is the time the  $k$ th rod takes to fall from the vertical to its collision with the  $(k+1)$ th rod.

In this notation,  $\Omega_k$  is the angular velocity of the  $k$ th rod at  $\theta_k = 0$  and  $\Omega_{fk}, \Omega_{bk}$  its angular velocities after it has fallen to  $\theta = \beta_1$ , just before and just after collision with the next rod, respectively.

**2.3. Study of the Two-Rod Collisions.** Now examine the collision between the  $k$ th and  $(k+1)$ th rods. Our assumptions guarantee that during the collision kinetic energy and angular momentum are conserved, and that while a rod falls its total energy (kinetic plus potential) is conserved. These conservation laws determine the solution. For the following argument, refer to Figure 2.1.

Just before the collision the  $k$ th rod has angular velocity  $\Omega_{fk}$  and the  $(k+1)$ th rod is at rest. After collision the  $k$ th rod has angular velocity  $\Omega_{bk}$  and the  $(k+1)$ th



**Fig. 2.1** The series of collisions between the  $(k-1)$ th,  $k$ th, and  $(k+1)$ th rods. In particular, the figure shows the state of the rods (a) just before the  $(k-1)$ th and  $k$ th rods collide; (b) just after the  $(k-1)$ th and  $k$ th rods have collided; (c) while the  $k$ th rod rotates towards the  $(k+1)$ th rod; (d) just before the  $k$ th and  $(k+1)$ th rods collide; (e) just after the  $k$ th and  $(k+1)$ th rods have collided; (f) while the  $(k+1)$ th rod rotates towards the  $(k+2)$ th rod.

rod has angular velocity  $\Omega_{k+1}$ . Applying conservation of kinetic energy,

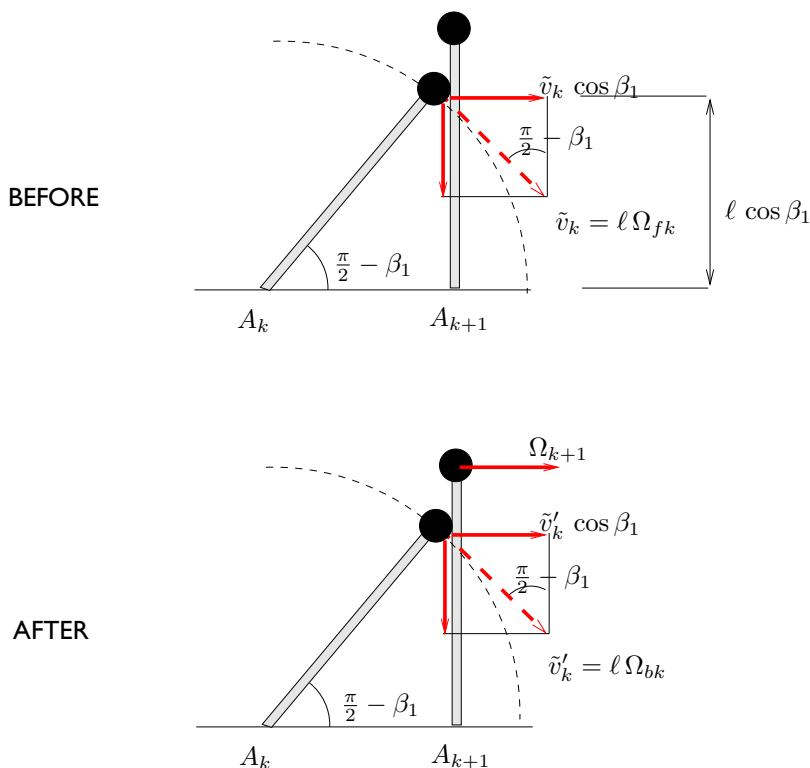
$$\frac{1}{2}I\Omega_{fk}^2 = \frac{1}{2}I\Omega_{bk}^2 + \frac{1}{2}I\Omega_{k+1}^2,$$

where  $I = m\ell^2$  is the moment of inertia of a rod. Therefore

$$(2.1) \quad \Omega_{fk}^2 = \Omega_{bk}^2 + \Omega_{k+1}^2.$$

Next we apply conservation of angular momentum with respect to point  $A_{k+1}$ . Just before the collision the  $k$ th rod has angular velocity  $\Omega_{fk}$  and thus translational velocity  $\tilde{v}_k = \ell\Omega_{fk}$ . From Figure 2.2 one can see that only the component  $\tilde{v}_k \cos \beta_1$  contributes to the angular momentum calculated around the point  $A_{k+1}$ , and that it does so with impact parameter  $\ell \cos \beta_1$ . The  $(k+1)$ th rod has no angular momentum initially. Therefore

$$L_{initial} = m(\tilde{v}_k \cos \beta_1)(\ell \cos \beta_1) = m\ell^2 \Omega_{fk} \cos^2 \beta_1.$$



**Fig. 2.2** The collision between the  $k$ th and  $(k + 1)$ th rods.

After the collision the  $(k + 1)$ th rod rotates around the point  $A_{k+1}$  with angular velocity  $\Omega_{k+1}$  and thus has angular momentum  $I\Omega_{k+1}$ . The  $k$ th rod has the new angular velocity  $\Omega_{bk}$ , again around the point  $A_k$ . Therefore it will contribute an angular momentum  $m \ell^2 \Omega_{bk} \cos^2 \beta_1$  with respect to  $A_{k+1}$ . Therefore

$$L_{final} = m \ell^2 \Omega_{bk} \cos^2 \beta_1 + I\Omega_{k+1} .$$

Conservation of angular momentum ( $L_{initial} = L_{final}$ ) yields

$$(2.2) \quad \Omega_{fk} \cos^2 \beta_1 = \Omega_{bk} \cos^2 \beta_1 + \Omega_{k+1} .$$

The system of equations (2.1) and (2.2) can be solved easily for  $\Omega_{k+1}$  and  $\Omega_{bk}$ :

$$(2.3) \quad \begin{aligned} \Omega_{k+1} &= f_+ \Omega_{fk} , \\ \Omega_{bk} &= \frac{\Omega_{k+1}}{f_-} , \end{aligned}$$

where

$$f_{\pm} \equiv \frac{2}{\cos^2 \beta_1 \pm 1 / \cos^2 \beta_1} .$$

Now consider the  $k$ th rod as it falls from the vertical to angle  $\beta_1$ , its position just before the collision. Conservation of total energy yields

$$\frac{1}{2} I \Omega_k^2 + m g \ell = \frac{1}{2} I \Omega_{fk}^2 + m g \ell \cos \beta_1 ,$$

or

$$(2.4) \quad \Omega_{fk}^2 = \Omega_k^2 + \frac{2g}{\ell} (1 - \cos \beta_1) .$$

Combining equations (2.3) and (2.4) we find

$$(2.5) \quad \Omega_{k+1}^2 = f_+^2 \Omega_k^2 + b ,$$

where

$$b = \frac{2g}{\ell} f_+^2 (1 - \cos \beta_1) .$$

Equation (2.5) is a mixed progression (i.e., a combination of an arithmetic and a geometric progression) and can be solved by well-known techniques (see Appendix A). The result is

$$\Omega_k^2 = f_+^{2(k-1)} \Omega_1^2 + b \frac{1 - f_+^{2(k-1)}}{1 - f_+^2} .$$

Recall that  $\Omega_1$  is the initial angular velocity of the first rod caused by the initial external push.

We show now that  $f_+ < 1$ . Since  $\beta_1 \neq 0, \pi/2$ ,  $x = \cos^2 \beta_1 \neq 1, 0$ . Then  $(x - 1/x)^2 > 0 \Rightarrow x^2 + 1/x^2 - 2 > 0 \Rightarrow x^2 + 1/x^2 > 2$ . From the last inequality it follows that  $f_+ = 2/(x^2 + 1/x^2) < 1$ .

Since  $f_+ < 1$  it follows that  $\lim_{n \rightarrow +\infty} f_+^n = 0$  and therefore

$$\lim_{k \rightarrow +\infty} \Omega_k^2 = \frac{2g}{\ell} (1 - \cos \beta_1) \frac{f_+^2}{1 - f_+^2} \equiv \Omega^2 .$$

Thus deep into the chain we find translational invariance: the initial angular velocity imparted to a rod by its neighbor becomes independent of position. Notice that in this limit the initial push given to the first rod becomes irrelevant.

**2.4. Wave Speed.** We can obtain the limiting speed of the wave by computing the time between collisions, working well into the chain where this becomes independent of position.

Apply conservation of energy for the  $n$ th rod as it begins moving and after it falls through an arbitrary angle  $\theta$ . This yields

$$\frac{1}{2} I \Omega_n^2 + mg\ell = \frac{1}{2} I \omega_n^2 + mg\ell \cos \theta .$$

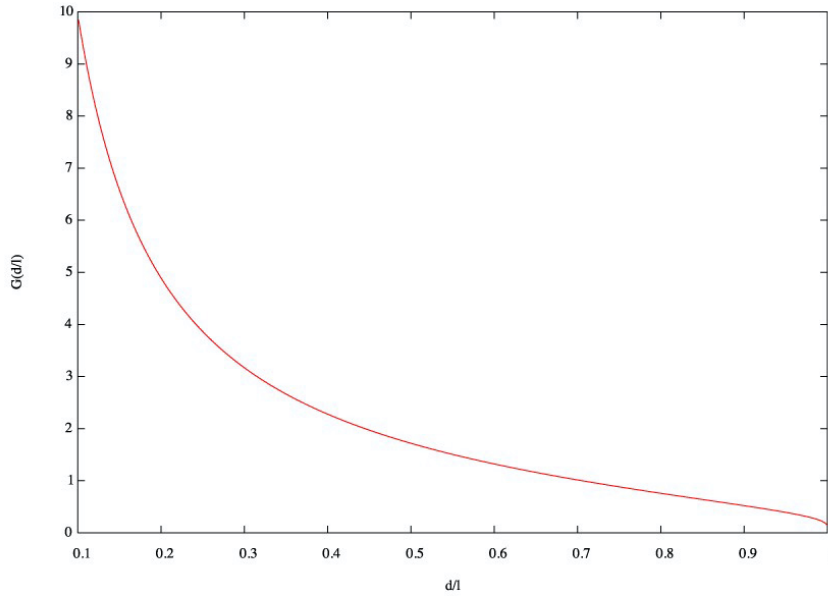
Setting  $\omega_n = d\theta/dt$ , we can separate  $t$  from  $\theta$  and solve for the time required for the rod to move from  $\theta = 0$  to  $\theta = \beta_1$ :

$$\int_0^{T_n} dt = \int_0^{\beta_1} \frac{d\theta}{\sqrt{\Omega_n^2 + \frac{2g}{\ell} - \frac{2g}{\ell} \cos \theta}} .$$

This integral can be expressed in terms of the complete elliptic integral of the first kind  $K(k)$  (see Appendix B):

$$T_n = \frac{2}{\sqrt{a_n + c}} \left[ K(k_n) - F\left(\frac{\pi - \beta_1}{2}, k_n\right) \right] ,$$

where  $a_n = \Omega_n^2 + \frac{2g}{\ell}$ ,  $c = \frac{2g}{\ell}$ , and  $k_n = \sqrt{\frac{2c}{a_n + c}}$ .



**Fig. 2.3** The scaling function  $G(d/\ell)$ .

In the limit of large  $n$ , the time  $T_n$  approaches a limiting value

$$T = \frac{2}{\sqrt{a+c}} \left[ K(k) - F\left(\frac{\pi - \beta_1}{2}, k\right) \right],$$

where  $a = \Omega^2 + \frac{2g}{\ell}$ ,  $c = \frac{2g}{\ell}$ , and  $k = \sqrt{\frac{2c}{a+c}}$ . The wave therefore approaches a limiting speed  $v = d/T$  given by

$$v = \frac{d}{2} \frac{\sqrt{a+c}}{K(k) - F\left(\frac{\pi - \beta_1}{2}, k\right)}.$$

A little algebra lets us write the wave speed in the scaling form of equation (1.1):

$$v = \sqrt{g\ell} G,$$

with

$$G\left(\frac{d}{\ell}\right) = \frac{d}{\ell} \frac{1}{k[K(k) - F\left(\frac{\pi - \beta_1}{2}, k\right)]}$$

and

$$k^2 = \frac{2(1 - f_+^2)}{(1 - \cos \beta_1)f_+^2 + 2(1 - f_+^2)}.$$

Since  $f_+$  and thus  $k$  depend only on  $\beta_1 = \sin^{-1}(d/\ell)$ , this  $G$  is indeed a function only of  $d/\ell$ , as required by scaling. The scaling function  $G$  is plotted in Figure 2.3.

No simple closed expression exists for complete elliptic integrals, but some insight into our solution comes from looking at the limit of very closely spaced rods ( $d \ll \ell$ ).

In this limit  $\beta_1 \approx d/\ell$  and  $f_+^2 \approx 1 - \beta_1^4$ . Using these in the above expression yields  $k^2 \approx 4\beta_1^2 \ll 1$ . Then

$$K(k) - F\left(\frac{\pi - \beta_1}{2}, k\right) = \int_{\frac{\pi - \beta_1}{2}}^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \approx \int_{\frac{\pi - \beta_1}{2}}^{\frac{\pi}{2}} dt = \frac{\beta_1}{2}.$$

From this we find

$$G\left(\frac{d}{\ell}\right) \approx \frac{1}{d/\ell}.$$

Thus the wave in very closely spaced rods moves very fast.

In a similar way one can examine the other extreme geometrical limit,  $d/\ell$  slightly smaller than unity. Put  $\beta_1 = \pi/2 - \sqrt{2}\epsilon$ . Then

$$\frac{d}{\ell} = \sin \beta_1 = \cos \sqrt{2}\epsilon \approx 1 - \epsilon,$$

while  $f_+ \approx 4\epsilon$  and  $k^2 \approx 1 - 16\epsilon^2$ . For  $k$  very near unity the complete elliptic integral is approximately  $K(k) \approx \ln(4/k')$ , where  $k'^2 = 1 - k^2$  [7]. Here this gives  $K(k) \approx \ln(1/\epsilon)$ , which diverges as  $\epsilon$  approaches zero. However, in this limit  $F$  is finite:

$$F\left(\frac{\pi - \beta_1}{2}, k\right) = \int_0^{\frac{\pi}{4} + \frac{\sqrt{\epsilon}}{2}} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \approx \int_0^{\frac{\pi}{4}} \frac{dt}{\cos t} = \ln(\sqrt{2} + 1).$$

In the limit  $d \approx \ell$  using the above approximations yields

$$v \approx \frac{\sqrt{g\ell}}{\ln\left[\frac{\ell}{(1+\sqrt{2})(\ell-d)}\right]},$$

or  $v \approx \sqrt{g\ell} G(d/\ell)$ , where

$$G\left(\frac{d}{\ell}\right) = -\frac{1}{\ln(1 + \sqrt{2}) + \ln(1 - \frac{d}{\ell})},$$

as  $d/\ell$  increases to 1. Thus as  $d$  gets very close to  $\ell$  the wave speed drops to zero. Physically this occurs because one rod gives a very small push to the next in line, which as a result takes a very long time to fall. The reader might wish to conduct a quick experiment to verify this conclusion or a more careful one to compare our theoretical results with experiment. Of course, the reader has noticed the consistency of our results with that of McLachlan et al. (1.1).

**3. Conclusions.** In this paper we have presented a set of assumptions for the propagation of the domino wave and we have computed the corresponding limiting speed. For simplicity we have presented the solution for a simplified geometry. However, the reader can easily transfer the solution to the case of dominoes with the shape of parallelepipeds—with appropriate adjustments of course.

**Appendix A. Mixed Progression.** Consider a sequence  $a_k$ ,  $k = 1, 2, \dots$ , with the recurrence relation

$$a_k = r a_{k-1} + b.$$



This is known as a mixed progression. We want to express  $a_k$  in terms of  $a_1$ ,  $r$ , and  $b$ . Multiply both sides by  $r^{n-k}$  and sum from  $k = 2$  to  $n$  (with  $n \geq 2$ ):

$$\sum_{k=2}^n r^{n-k} a_k = \sum_{k=2}^n (r^{n-k+1} a_{k-1} + r^{n-k} b).$$

In the first term on the right-hand side replace  $k$  by  $k' = k - 1$  and in the second replace  $k$  by  $k' = n - k$ . This yields

$$\sum_{k=2}^n r^{n-k} a_k = \sum_{k'=1}^{n-1} r^{n-k'} a_{k'} + b \sum_{k'=0}^{n-2} r^{k'}.$$

The first two sums have nearly all terms in common (all but the  $n$ th on the left and the first on the right). Canceling the terms in common and evaluating the third sum yields the desired solution:

$$a_n = r^{n-1} a_1 + b \frac{1 - r^{n-1}}{1 - r}.$$

**Appendix B. The Elliptic Integral of First Kind.** The elliptic integral of the first kind [7, 8] is defined by

$$F(\phi_0, k) = \int_0^{\phi_0} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad 0 \leq k < 1.$$

When  $\phi_0 = \pi/2$  this is called the complete elliptic integral of the first kind, denoted by  $K(k)$ :

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

The integral

$$I(\theta_0) = \int_0^{\theta_0} \frac{d\theta}{\sqrt{a - c \cos \theta}}, \quad 0 < c < a,$$

can be expressed in terms of the elliptic integral of the first kind as follows. First make the change of variable  $\theta = \pi - 2t$ :

$$I(\theta_0) = 2 \int_{\pi/2 - \theta_0/2}^{\theta_0/2} \frac{dt}{\sqrt{a + c \cos(2t)}}.$$

Using the identity  $\cos 2t = 1 - 2 \sin^2 t$  this becomes:

$$\begin{aligned} I(\theta_0) &= 2 \int_{\pi/2 - \theta_0/2}^{\pi/2} \frac{dt}{\sqrt{(a + c) - 2c \sin^2 t}} \\ &= \frac{2}{\sqrt{a + c}} \int_{\pi/2 - \theta_0/2}^{\pi/2} \frac{dt}{\sqrt{1 - \frac{2c}{a+c} \sin^2 t}}. \end{aligned}$$

Finally we set

$$k^2 \equiv \frac{2c}{a+c},$$

and we rewrite the above result in the form

$$\begin{aligned} I(\theta_0) &= \frac{2}{\sqrt{a+c}} \left( \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2 \sin^2 t}} - \int_0^{\pi/2-\theta_0/2} \frac{dt}{\sqrt{1-k^2 \sin^2 t}} \right) \\ &= \frac{2}{\sqrt{a+c}} \left[ K(k) - F\left(\frac{\pi-\theta_0}{2}, k\right) \right]. \end{aligned}$$

**Acknowledgments.** We thank the referees for bringing to our attention the articles of Walker [3] and Shaw [6] and the book of Banks [5].

**Note Added in Proof.** After this work was completed a paper [9] with somewhat similar analysis appeared on the Cornell archives.

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