Variational principles

and

Optimality in nature

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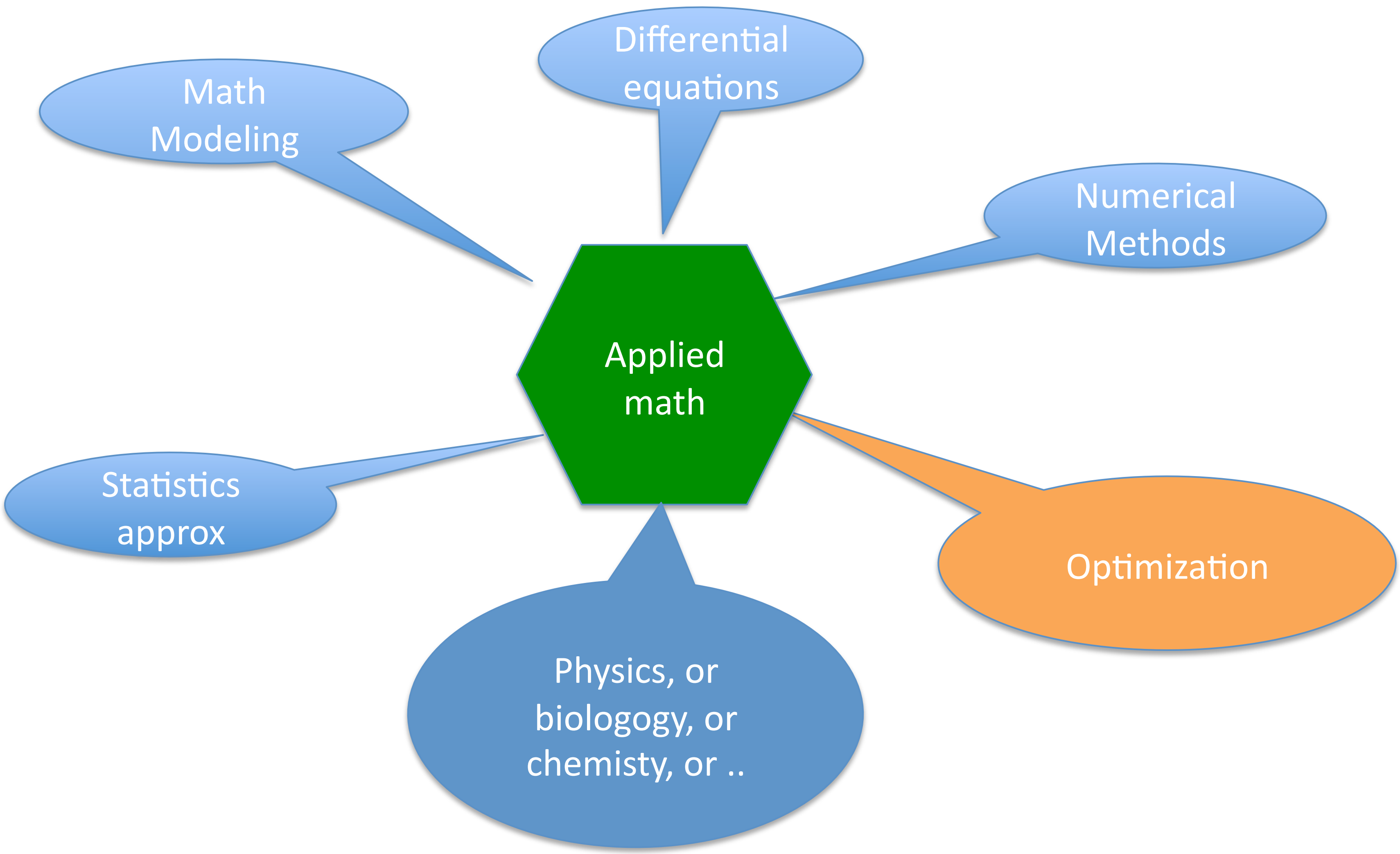
Components of applied math



Why study applied math?

- Applied Math is one of the most prestigious and favored profession. Everyone needs applied math.

- The applied math problems are inspired by a nature phenomenon or a social need. The phenomena is modeled and rigorously investigated. This work combines social value, elegance and beauty.

- Study of applied problems lead to development of calculus, linear algebra, geometry, probability, etc.

- And, yes, we can also teach. In academia, we work on our own problems and collaborate with our colleagues in departments of science, engineering, mining, finances, and medicine.

Optimality

The desire for optimality (perfection) is inherent for human race. The search for extremes inspires mountaineers, scientists, mathematicians, and the rest of the human race.

Dante:
All that is superfluous displeases God and Nature
All that displeases God and Nature is evil.

In engineering, optimal projects are considered beautiful and rational, and the far-from-optimal ones are called ugly and meaningless. Obviously, every engineer tries to create the best project and he/she relies on optimization methods to achieve the goal.

Two types of optimization problems:

1. Find an optimal solution (engineering)
2. Understand optimality of a natural phenomenon (science)

Beginning: Geometry
Isoperimetric Problem (Dido Problem)

- Probably the first variational problem (isoperimetric problem) has been solved by wise Dido, founder and queen of Carthage the mighty rival of Rome:
- **What is the area of land of maximum area that can be encircled by a rope of given length?**
- ***Four hinges proof by Jacob Steiner***

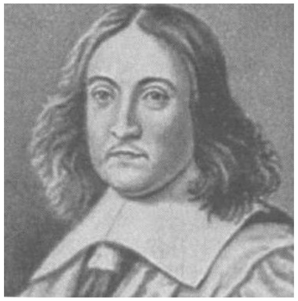
- 1. The domain is convex
- 2. In the perimeter is cut in half, the area is cut too.
- 3. The hinge argument

Search for optimality in Nature

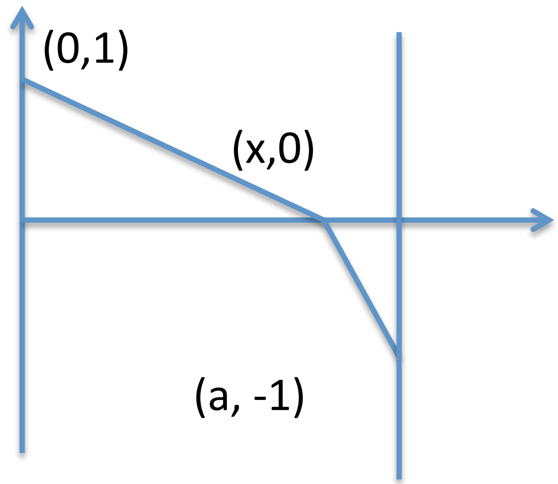
- The general principle by Pierre Louis Maupertuis (1698-1759) proclaims: *If there occur some changes in nature, the amount of action necessary for this change must be as small as possible.* This principle proclaims that the nature always finds the "best" way to reach a goal."

- Leonhard Euler: "Since the fabric of the Universe is most perfect and is the work of a most wise Creator, nothing whatsoever takes place in the Universe in which some relation of maximum and minimum does not appear."

Ferma's principle



Fermat's *principle of least time*
(which he used to derive Snell's law in 1657)
was the first variational principle in physics



$$T = \frac{\sqrt{1+x^2}}{v_1} + \frac{\sqrt{1+(x-a)^2}}{v_2};$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{1+x^2}} + \frac{x-a}{v_2\sqrt{1+(x-a)^2}} = \boxed{\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0}$$

Generally, when $v = v(y)$:

$$v = \frac{ds}{dt}; \quad dt = \frac{ds}{v}; \quad ds^2 = dy^2 + dx^2; \quad ds = \sqrt{1+y'^2}dx$$

$$T = \min_{y(x)} \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}dx}{v(y)} \text{ s.t. } y(x_0) = y_{0,}\ y(x_1) = y_{1,}$$

Brachistochrone

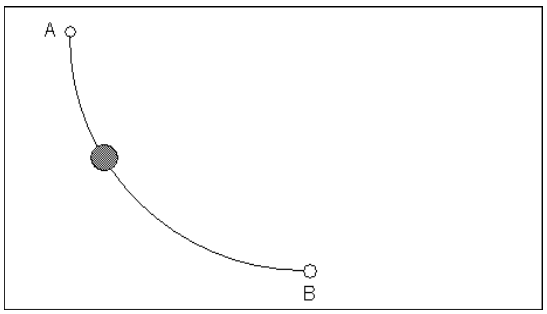
Johann Bernulli challenge: (June 1696): **Find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time.**

… it is known with certainty that there is scarcely anything which more greatly excites noble and ingenious spirits to labors which lead to the increase of knowledge than to propose difficult and at the same time useful problems through the solution of which, as by no other means, they may attain to fame and build for themselves eternal monuments among posterity.

- Five solutions were obtained by: Johann Bernulli himself, Leibniz, L'Hospital, Newton, and Jacob Bernoulli.

Brachistochrone

- Brachistochrone is the same Fermat problem, but the speed is defined from the conservation of energy law:



$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$

$$T = \min_{y(x)} \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}dx}{v(y)} = \min_{y(x)} \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}dx}{\sqrt{2gy}}$$

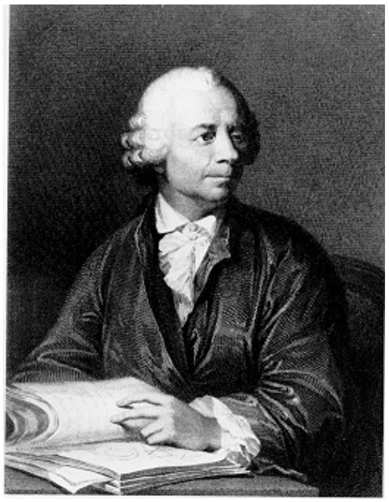
$$\text{s.t. } y(x_0) = y_{0,}, y(x_1) = y_{1,}$$

Solution: an inverted cycloid

$$x = C(\theta - \sin\theta) + C_1,$$

$$y = C(1 - \cos\theta)$$

Euler equation (functional derivative)



$$J = \int_a^b F(t,u,u')dt, \quad \delta J = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \right) dt + o(|\delta u)|) =$$

$$\int_a^b \left(\frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial u'} \right) \delta u dt + \left[\frac{\partial F}{\partial u'}\delta u \right]_a^b + o(|\delta u)|).$$

$$J \to \min_{u(x)} \Rightarrow \delta J \geq 0 \ \forall \delta u \Rightarrow$$

$$S(u,u',u'') = \left(\frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial u'} \right) = 0, x \in (a,b)$$

S(u,u’,u’’) is the functional derivative, an analog of gradient for the functional argument.

Using this equation, one can derive solution to Brachistochrone problem, find minimal surface of revolution, solve Dido problem. etc.

Lagrangian

- Lagrange applied Euler equation to reformulate Newton law of motion:

$$mu'' = f \qquad (*)$$

$$\delta\left(\int_a^b \frac{1}{2} m\left(u'\right)^2 dx\right) = (mu'')\delta u$$

$$\delta\left(\int_a^b V(u)dx\right) = V'\delta u \qquad (f = V')$$

$$(*) \Rightarrow \delta\left(\int_a^b (K - V)dx\right) = 0$$

$$K = \frac{1}{2} m\left(u'\right)^2$$

Why minus?

Lagrange, Mécanique analytique

"The admirers of the Analysis will be pleased to learn that Mechanics became one of its new branches”

The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasoning: but only algebraic operations, subject to a regular and uniform rule of procedure.

Lagrangian

Examples:

Oscillator

$$mu'' + cu = 0 \Leftrightarrow \delta \int (K - V) dx = 0$$

$$K = \frac{1}{2} m(u')^2, \quad V = \frac{1}{2} c(u)^2$$

Celestial mechanics

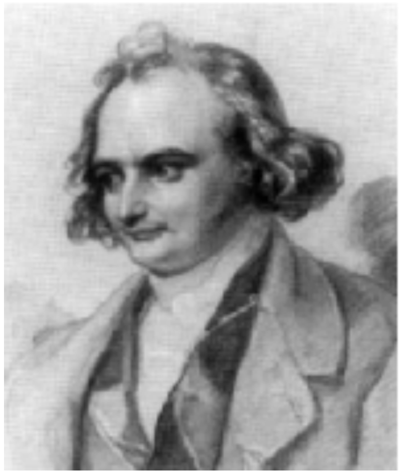
$$K = \frac{1}{2} \sum_i m_i (\rho_i')^2,$$

$$V = \frac{1}{2} \sum_{i,j} \frac{G m_i m_j}{|\rho_i - \rho_j|}$$

$$\rho_i'' = \sum_{i,j} \frac{G m_j}{|\rho_i - \rho_j|^3} (\rho_i - \rho_j)$$

Is Newton law of motion equivalent to minimization of action K-V ?

If so, then our world is indeed "the best of all possible worlds" according to Gottfried Leibniz (1710)



Oops! Jacobi variation.

- The Euler equation is the stationarity condition. Does it correspond to a local minimum?
- Look at second variation. Assume that S(u)=0, then

$$\delta^2 J = \int_a^b \left(\left[\frac{\partial^2 F}{\partial u^2} \right] (\delta u)^2 + \left[\frac{\partial^2 F}{\partial u'^2} \right] (\delta u')^2 + \frac{\partial^2 F}{\partial u' \partial u} \delta u' \partial u \right) dt + o(|\delta u)^2 |)$$

$$J \to \min_{u(x)} \Rightarrow \delta J = 0, \quad \delta^2 J \geq 0 \ \forall \delta u$$

Example

$$K = \frac{1}{2} m (u')^2, \quad V = \frac{1}{2} c (u)^2$$

$$\delta^2 J = \int_0^b \left(m (\delta u')^2 - c (\delta u)^2 \right) dt$$

For long trajectories, the second variation changes its sign! (Newton law is an instant relation_.

$$\text{Take } \delta u = \varepsilon\, t (b - t). \quad \delta^2 J = \varepsilon \frac{b^3 c}{30} \left(b^2 - 10 \frac{m}{c} \right), \quad \delta^2 J < 0 \text{ if } b > \sqrt{\frac{10m}{c}}$$

Saving the natural minimal principle through unnatural geometry

- In Minkowski geometry, the time axis is imaginary, the distance is computed as

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2$$

- If we replace *t = i* τ, kinetic energy changes its sign, *T-> -T* and action becomes the negative of sum of kinetic and potential energies.

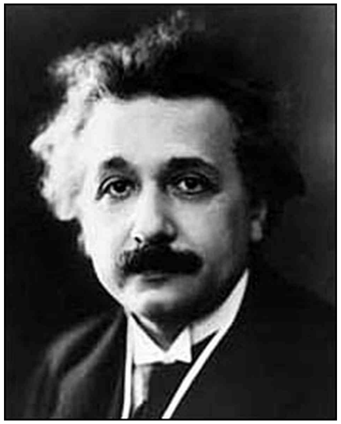
$$\frac{d}{dt} = \frac{d}{id\tau}, \quad K = -\frac{m}{2}\left(\frac{du}{d\tau}\right)^2, \quad L = -(K+V)$$

- *L* is still real because the of double differentiation.
- The stationary equations are the same, but the energy in Minkovsky geometry reaches its maximum.

Example:
An oscillator

$$L = -\frac{m}{2}\left(\frac{du}{d\tau}\right)^2 - \frac{c}{2}u^2 \quad S(u) = m\left(\frac{du^2}{d\tau^2}\right) + cu = 0; \quad -m\left(\frac{du^2}{dt}\right) + cu = 0$$

$$\delta^2 J = -\int_a^b \left(c(\delta u)^2 + m(\delta u')^2\right)dt \le 0 \text{ (max)}$$



Albert Einstein

STR solves the mystery

According to Special Relativity (1905), a particle moves so that its integral of its whole energy is maximized

$$(*) \quad W = \int_{t_0}^{t_f} (V(u) + mc^2)dt'$$

- In the coordinate system of an observer, the time is different, than the time of a moving particle

$$dt' = \sqrt{1 - v^2/c^2}\; dt \cong \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)dt$$

- Now, substitute this into (*) and find:

$$(V(u) + mc^2)dt' = dt \cong (V(u) + mc^2)\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)dt \cong$$

$$\left[mc^2 + V(u) - \frac{1}{2}mv^2 + O(1/c^2)\right]dt$$

- So, the term *-K* (including the minus sign!) appears because of the difference in time of a particle and of an observer!

Other (contemporary) variational principles in continua

A stable equilibrium is always a local minimum of the energy

Systems with many equilibria lead to a chaotic motion or jerky transitions between them (proteins, crash of a construction).

- The principle of maximal energy release rate in cracking
- The principle of maximal energy dissipation
- Minimum entropy principle in chemical reactions

- Biological principles:

The mathematical method: evolutionary games that model animal or human behavior models, establishing their evolutionary usefulness

Optimality of evolution

R. A. Fisher's **Fundamental theorem of natural selection,**
in *Genetical theory of natural selection (1930)*

The rate of increase in fitness of any organism at any time is equal to its genetic variance in fitness at that time. A living being maximizes *Inclusive fitness* – the probability to send its genes to the next generation

Fisher's example:

- Between two groups (male and female) of different sizes *s1* and *s2*, *s1 < s2*, the probabilities *p1* and *p2* to find a mate from the opposite group are, respectively:

$$p_1 = \frac{s_2}{s_1 + s_2}, \quad p_2 = \frac{s_1}{s_1 + s_2}, \quad p_1 > p_{2.}$$

The smaller group *s1* is in a better position, therefore, its size increases, and is grows faster. The equilibrium is reached when *s1=s2.*

- *Why the election results are always close?*

Doves and hawks

”The logic of animal conflict” by Maynard, Smith and Price 1973

- Suppose there are C per cent of doves and (100-C) per cent of hawks
- When a dove meets a dove, they split bounty *V* equally, each gets *V/2*
- When a dove meets a hawk, the hawk takes all: a dove gets 0, a hawk gets *V.*
- When a hawk meets a hawk, they fight and each hawk looses *P*.

- The average gain of a dove is V*C/2.*
- The average gain of a hawk is *VC -P(1- C).*

Equilibrium

$$V\frac{C}{2} = VC - P(1-C)$$

$$C = \frac{2P}{3V+2P}, \quad 1-C = \frac{3V}{3V+2P}$$

All things are crated optimal
The question is: in what sense?

The symmetry of trees correspond to the equally high strength against the "worst direction" of the wind.

The symmetry solves the problem: maximize the strength in the weakest direction.

Mistery of spriral trees

math.utah.edu/~cherk/spiral-trees/story.html

- Why do they spiral?
- Postulate: *If a natural design becomes more complex, there is a reason behind.*
- Problem: find the goal functional that reaches its minimum at realistic angles (about 45^o of spiraling).
- The answer is not unique. Spiraling allow to keep the symmetry when roots at one side die. It is a compromise between the length of the path along the thread from a root to a brunch, and flexibility due to spiraling.

- S. Leelavanishkul and A. Ch. 2004, 2009

Novel branches of variational calculus
Problems without solutions

- infinitely oscillating minimizing sequence (Laurence Young, 1905-2000)

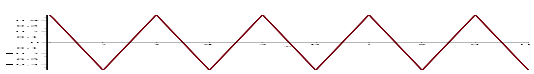
$$J = \inf_{u(x):u(0)=0,u(1)=0} R(u), \quad R(u) = \int_0^1 \left[\left((u')^2 - 1\right)^2 + u^2\right] dx$$

$R(u) \geq 0 \ \forall u;$ find a minimizing sequence $\hat{u}_s : \lim J(\hat{u}_s) = 0.$

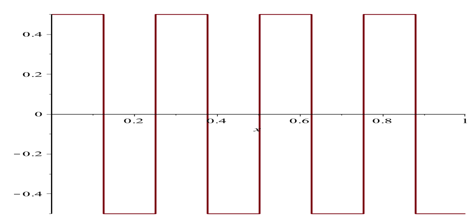
$$1.\ |\hat{u}_s'| = 1 \Rightarrow R(\hat{u}_s) = \int_0^1 \left[u_s^{\ 2}\right] dx$$

$$2.\ \hat{u}_s(x) = \begin{cases} x, & \text{if } x \in \left[0, \frac{1}{2s}\right) \\ \frac{1}{2s} - x, & \text{if } x \in \left[\frac{1}{2s}, \frac{1}{s}\right) \end{cases}, \quad \hat{u}_s(x) \text{ is } \frac{1}{s} \text{ - periodic}$$

u(x)



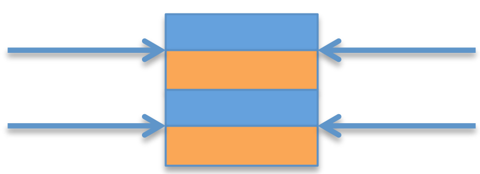
u′(x)



Multidimensional problem: Optimal composite

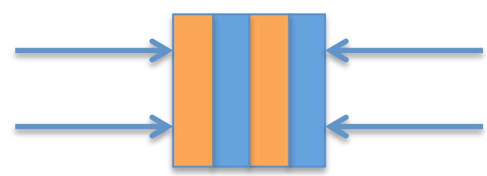
Problem: Find materials' layout that minimizes the energy of the system (the work of applied forces) plus the total cost of materials.

Feature: Infinite fast alternations, the faster the better. This time, the solution also depends on geometrical shape of mixing the subdomains of pure materials, because of boundary conditions: normal stress and tangent deformation stay continuous at the boundary between several materials:



Maximal stiffness

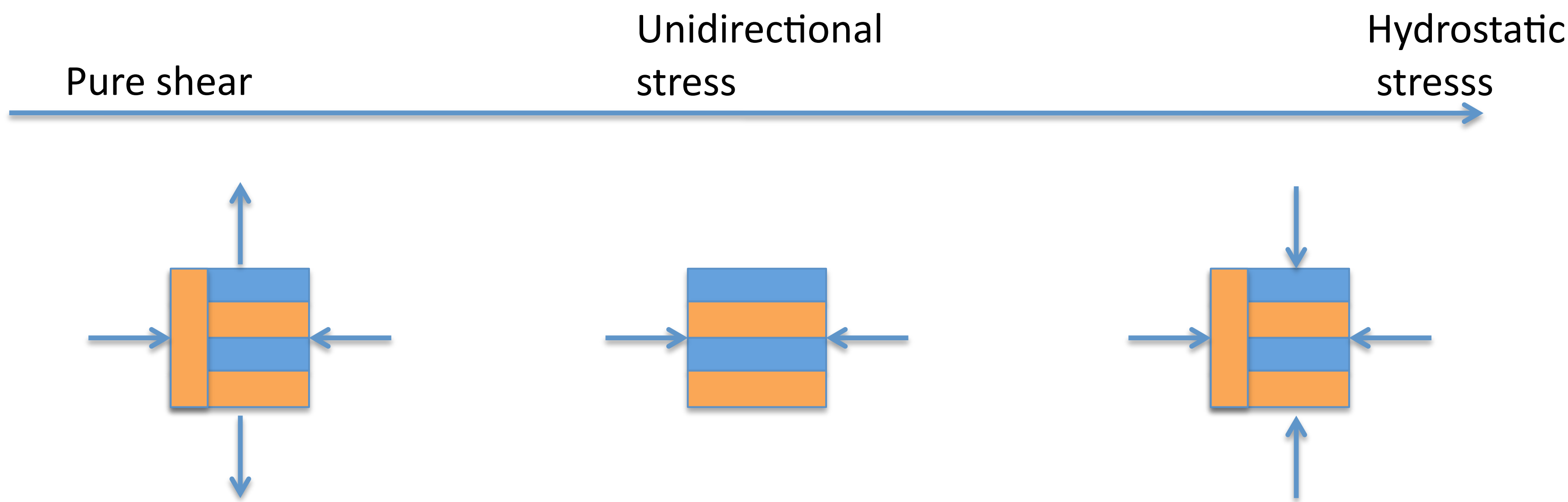
$$C_{eff} = m_1 C_1 + m_2 C_2$$



Minimal stiffness

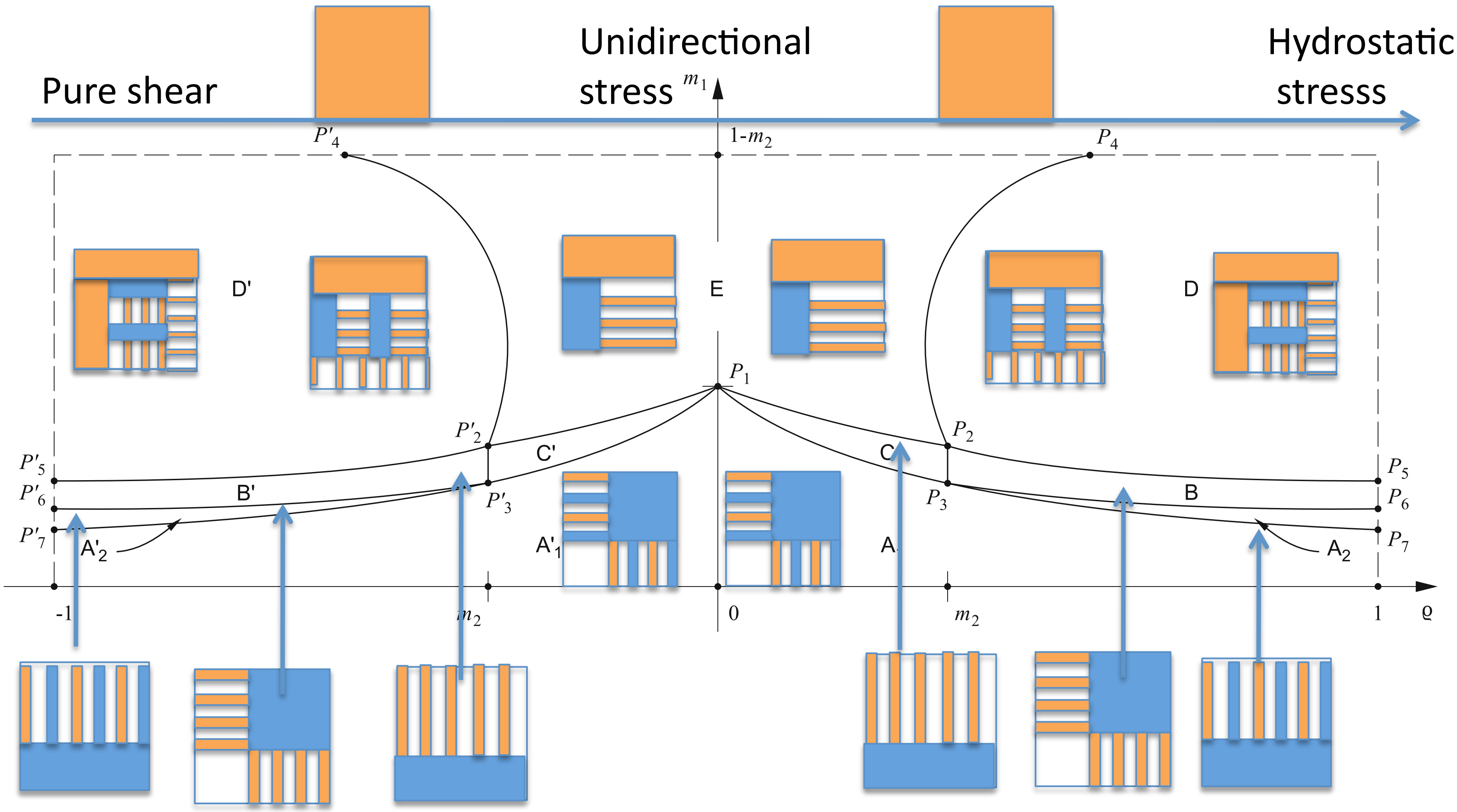
$$C_{eff} = \left(\frac{m_1}{C_1} + \frac{m_2}{C_2} \right)^{-1}$$

Two-material optimal structures adapt to an anisotropy of a loading



Color code:
orange – strong material
blue -- weak material,
white -- void.

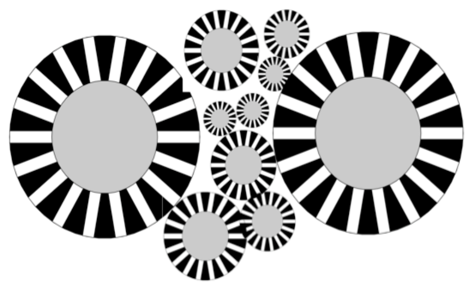
Three-material optimal structures are adapted to an anisotropy and volume fraction m1



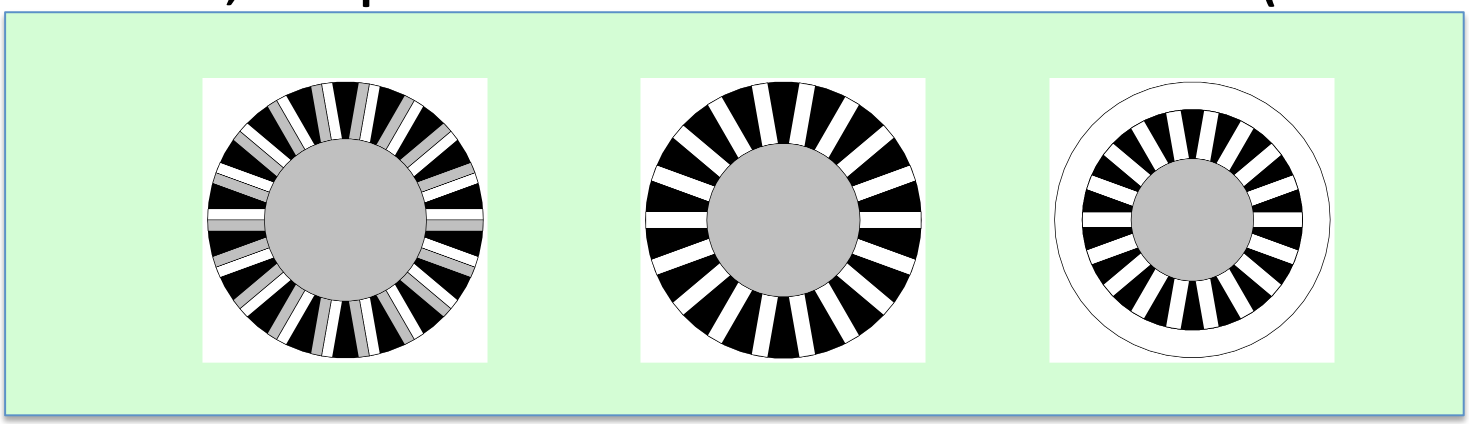
Ch & Zhang, 2011, Ch & Dzierżanowski, 2013

Optimal geometries are not unique. Isotropic wheel assemblages

Hashin-Shtrikman type assembly



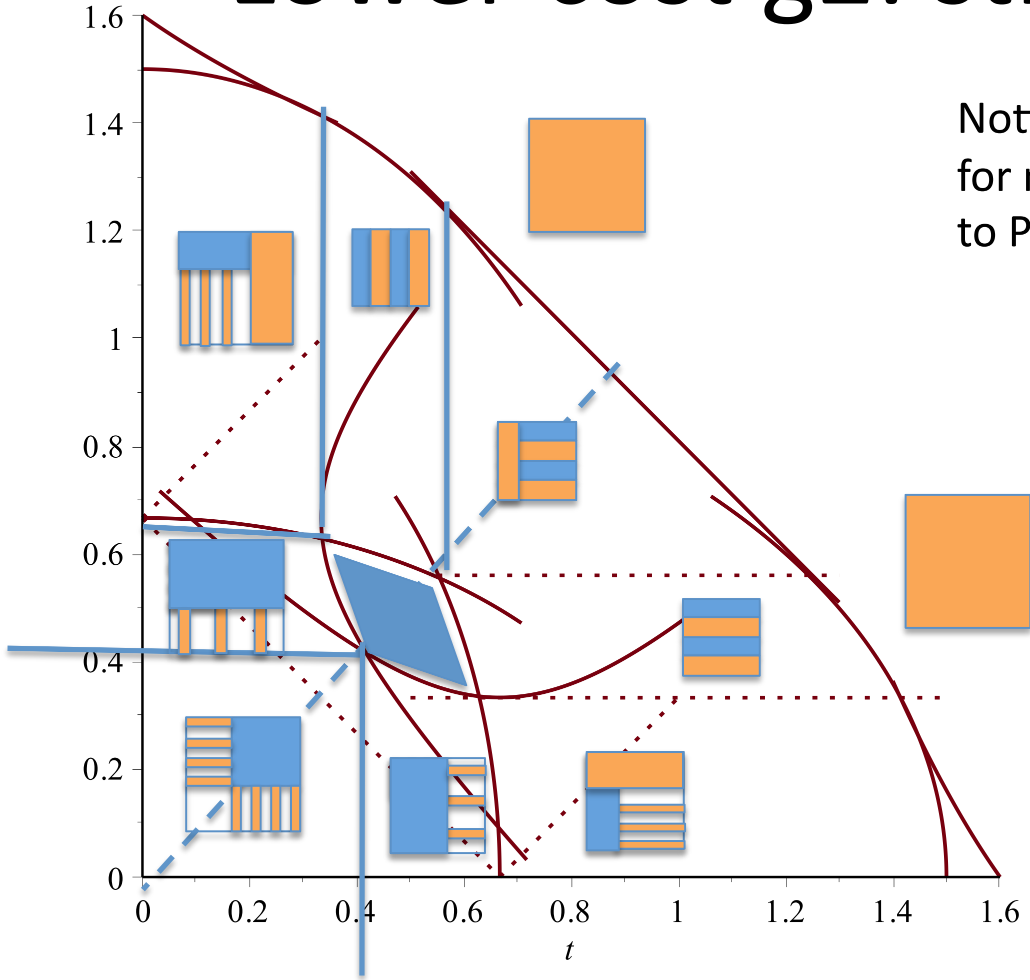
Elements, in dependence on volume fraction of the best (white field) material.



strong – white, intermediate – gray, void – black.

A.Cherkaev, 2011

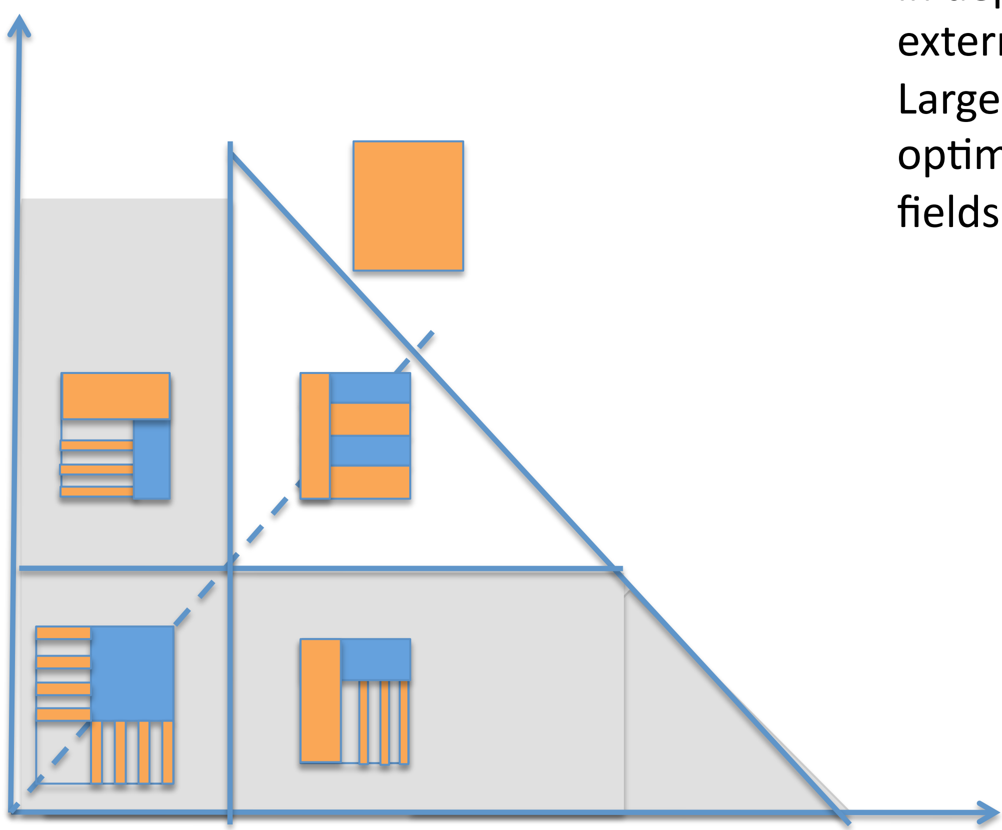
Lower cost g2: structures



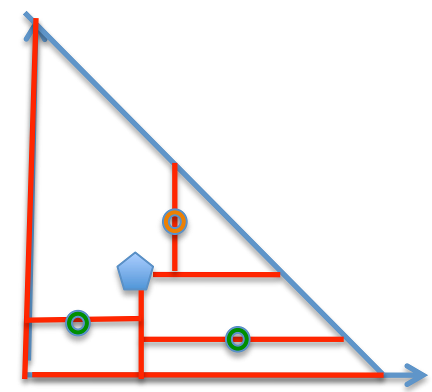
Notice that the optimal structures for more intensive fields correspond to P1-P2 composites.

After the first transition, P1 always enters the 1-2-3 structures (gray areas) via 1-3 laminate.

Special case of the materials' cost

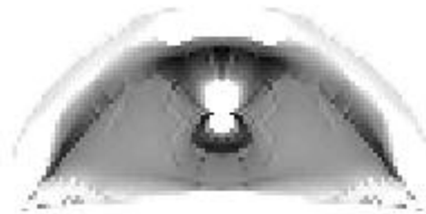


The graph shows the optimal structures in dependence of eigenvalues of the external stress field.
Large stresses correspond to the optimality of pure stiff material. Gray fields show three–material structures.



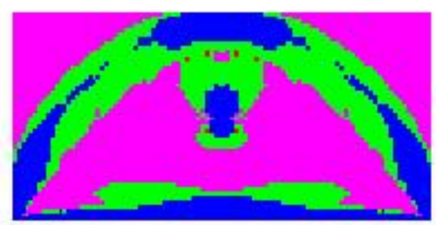
Example of three-material design (Computations by Nathan Briggs) Bridge #1

Black-and white design volume fraction



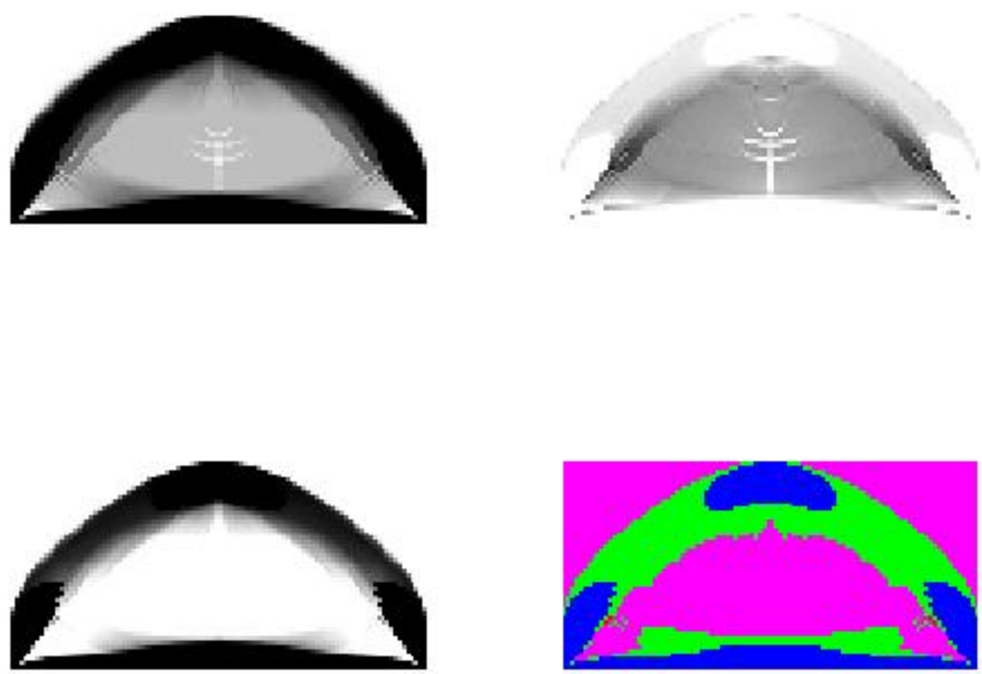
Volume fraction of P2

Volume fraction of P1

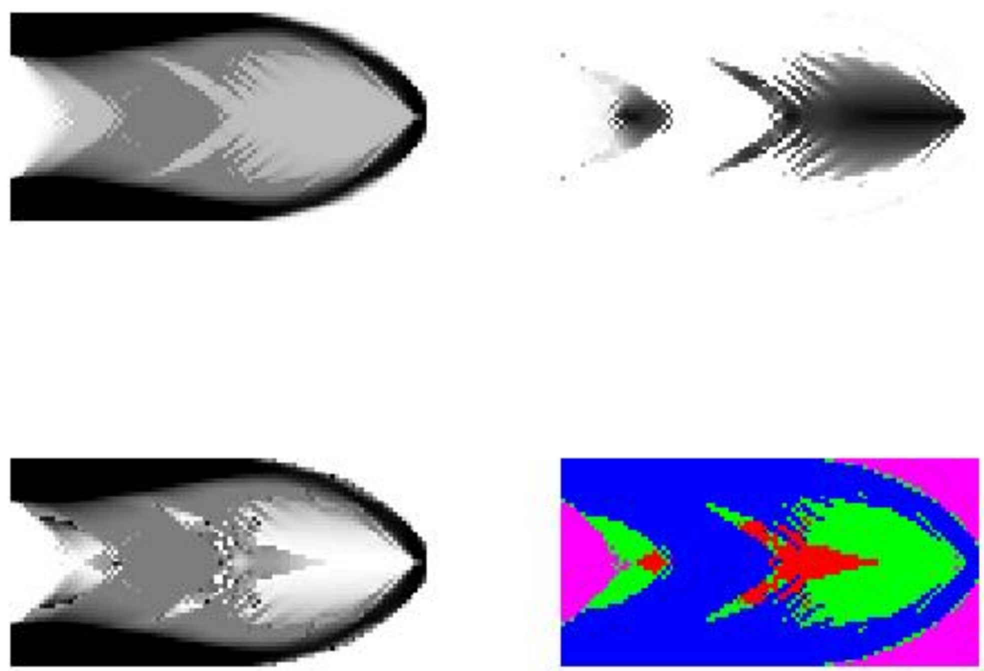


Distribution of composites

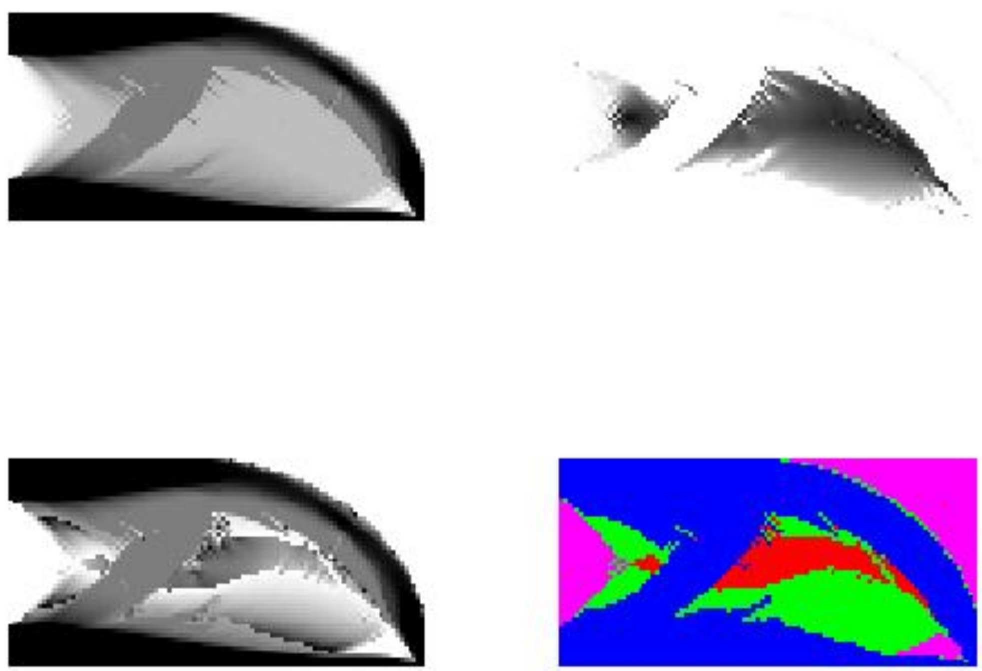
Bridge #2, smaller total weight



Cantilever, loaded at the middle

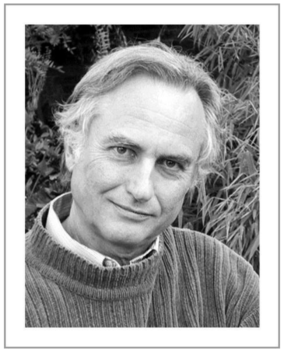


Cantilever, loaded at the the edge



Thank you!

After physics, optimality of evolution



Instead of action, a living being maximizes *Inclusive fitness* – the probability to send its genes to the next generation.

Selfish Gene by Richard Dawkins 1976.

The mathematical method: evolutionary games.

- Example:
- Between two groups (male and female) of different sizes *s1* and *s2*, *s1* < *s2*, the probabilities *p1* and *p2* to find a mate from the opposite group are:

$$p_1 = \frac{s_2}{s_1 + s_2}, \quad p_2 = \frac{s_1}{s_1 + s_2}, \quad p_1 > p_{2.}$$

- The smaller group is in a better position and its size increases, so is grows faster. The equilibrium is reached when *s1=s2.*
-
- *Why the election results are always close?*

Plan

- Introduction: Extremal principles: Description through minimization. Dido problem.
- History
  - Fermat principle.
  - Calculus of variations. Euler equation, Brachistochrone
  - Lagrange approch
  - Jacobi Columbus principle.
  - Special relativity. Minkowski geometry.
- Today:
  - Rationality of the evolution.
  - Male and Female newborns. Evolutionary games
  - Optimality of the spiral tree
- Problems without solution.
  - Minimal surface
  - much ado about nothing.