Remarks on Convexity

March 22, 2019

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Figure 1: Convexity

1 Convexity: Scalar dunction

1.1 Convex function of a scalar argument

Convex function A real-valued continuous function $f(x)$ of a scalar argument $x \in [a, b]$ is convex on an interval if for any two points x_1 and x_2 in $[a, b]$ and any t, $0 < t < 1$, the inequality holds

$$f(m\,x_1 + (1-m)x_2] \le m\,f(x_1) + (1-m)f(x_2) \quad \forall m \in (0,1) \tag{1}$$

which states that the graph of $f(x)$ lies below the secant line between x_1 and x_2.

Convexity in a point Another useful form of that inequality defines convexity in a point $x \in R$. If we rename the arguments in (1) as:

$$x = m\,x_1 + (1-m)x_2, \quad v_1 = x_1 - x, \quad v_2 = x_2 - x$$

inequality (1) will define the convexity of $f(x)$, $x \in R$ in the point x:

$$f(x) \le m_1 f(x+v_1) + m_2 f(x+v_2) \quad \forall m_1, m_2, v_1, v_2 \quad \text{such that}$$
$$m_1 \ge 0,\ m_2 \ge 0, \quad m_1 + m_2 = 1, \quad m_1 v_1 + m_2 v_2 = 0, \tag{2}$$

Here v_1 and v_2 are any perturbations of the argument x with zero mean value, see (2).

Example 1.1 Function $f(x) = \left(x^2 - 1\right)^2$ is convex in all points outside the interval $x \notin [-1, 1]$ and is not convex inside this interval.

Jensen's inequality, integral form The definition of convexity (2) in the point x is extended to the Jensen inequality for a function $f(x)$, $x \in R$

$$f(x) \le \sum_{i=1}^{n} m_i f(x+v_i), \quad \forall m_i, v_i, \quad i = 1, \dots n, \quad \text{such that} \tag{3}$$

$$m_i \ge 0, \quad \sum_{i=1}^{n} m_i = 1, \quad \sum_{i=1}^{n} m_i v_i = 0 \tag{4}$$

We define convexity of an integrable function $f(x)$ where $x \in [a, b]$ at the point $x = A$. Jensen inequality is naturally extended to the statement: Function an integrable function $f(x)$ is convex, if the inequality holds

$$f(A) \le \frac{1}{b-a} \int_a^b f(A + v(x))dx \quad \forall v(x) \quad \text{such that} \int_a^b v(x)dx = 0 \tag{5}$$

Here, function $v(x)$ is a perturbation with zero mean value. The relation (5) says that any perturbation $v(x)$ with zero mean value does not decrease the value of the integral.

Perturbation of a strictly convex function Function $f(x)$ is strictly convex, if the equality

$$f(A) = \frac{1}{b-a}\int_a^b f(A+v(x))dx \qquad (6)$$

implies that $v(x) = 0$. In other words, any nonzero perturbation with zero mean value increases the value of the integral.

Example 1.2 Function $f(x) = x^2$ is strictly convex because

$$\frac{1}{b-a}\int_a^b (A+v(x))^2 dx$$
$$= A^2 + \frac{2A}{b-a}\int_a^b v(x)\,dx \quad + \frac{1}{b-a}\int_a^b v^2(x)dx > A^2 \quad \text{if} \quad \int_a^b v^2(x)dx \neq 0$$

Notice, that the second term in the right-hand side in the above sum is zero because the mean value of v is zero and the third term is positive if $v(x) \neq 0$.

Example 1.3 Affine function $f(x) = c\,x + d$ is convex but not strictly convex because

$$\frac{1}{b-a}\int_a^b (c(A+v(x)) + d)\,dx$$
$$= cA + d + \frac{c}{b-a}\int_a^b v(x)\,dx \quad = cA + d = f(A)$$

Example 1.4 Function $f(x) = |x|$ is convex everywhere but it is not strictly convex if $x \neq 0$. At the point $x = 0$, it is strictly convex.

1.2 Convex envelope

Assume that a differentiable function $f(x)$ grows superlinearly and is bounded from below

$$\lim_{|x|\to\infty} \frac{f(x)}{|x|} = \infty, \quad \exists c: \ f(x) \geq c \quad \forall x \in R$$

Assume also that $f(x)$ is nonconvex and inequality (5) is not valid. There exist perturbations that make the integral in the right-hand side of (5) smaller than $f(A)$. A natural question arises: Find a perturbation $v(x)$ with zero mean that delivers minimum of this integral

$$\mathcal{C}f(A) = \min_{v(x)} \frac{1}{b-a}\int_a^b f(A+v(x))dx \quad \text{subject to} \ : \quad \int_a^b v(x)dx = 0 \qquad (7)$$

This minimum $\mathcal{C}f(A)$ is called the *convex envelope* of $f(A)$.

Figure 2: Convex envelope

Since that $f(x)$ is differentiable and grows faster than a linear function, the optimal $v(x)$ is finite and satisfies the equation:

$$\frac{d}{dv}(f(A+v(x))+\lambda v(x))=0$$

where λ is the Lagrange multiplier by the integral constraint in (7).

The optimal values of $v(x)$ must have a common derivative:

$$\frac{d}{dv}f(A+v_i)=\lambda \quad i=1,2;$$

It is geometrically clear that optimal perturbation $v(x)$ is piece-wise constant and takes no more than two values. Convex envelope $\mathcal{C}f(x)$ of $f(x)$ either coincides with $f(x)$ or is a linear function on an interval of non-convexity; it is supported by two boundary points v_1, v_2 of this interval that have the same derivative. Because the component of the convex envelope being a linear function is supported by two points, optimal $v(x)$ takes no more than two values. The convex envelope is defined as

$$\mathcal{C}f(x)=\min_{v_1,v_2,t}\left(m_1\, f(x+v_1)+m_2 f(x+v_2)\right), \tag{8}$$

$$m_1\geq 0, \quad m_2\geq 0, \quad m_1+m_2=1, \quad m_1v_1+m_2v_2=0 \tag{9}$$

in the points x where $f(x)$ is convex, $m_1=1$, $m_2=0$ and the convex envelope $\mathcal{C}f(x)$ coincides with the function itself, $\mathcal{C}f(x)=f(x)$.

Properties of convex envelopes

- The derivative $\frac{d}{dx}\mathcal{C}f(x)$ of $\mathcal{C}f(x)$ monotonically increases; it coincides with $f'(x)$ in the intervals where $f(x)$ is convex and is constant in the intervals of non-convexity of $f(x)$.
- The second derivative $\frac{d^2}{dx^2}\mathcal{C}f(x)$ is nonnegative; it is equal to zero in the interval of non-convexity.
- One can show that $\mathcal{C}f(x)$ is the maximal convex function that is smaller than or equal to $f(x)$ in each point x.

1.3 Examples

Example 1.5 Function

$$f(x)=\left(x^2-1\right)^2, \quad x\in R$$

is not convex in the interval $(-1,1)$, and is convex outside of this interval. Convex envelope of function $f(x)$ is

$$\mathcal{C}f(x)=\begin{cases} f(x) & |x|\geq 1 \\ 0 & |x|<1 \end{cases}$$

the supporting points are $x_{1,2} = \pm 1$. In these points, the function and its derivative coincide with the convex envelope and its derivative, respectively, $\mathcal{C}f(x)|_{x=\pm 1} = f(x)|_{x=\pm 1}$ and $\mathcal{C}f'(x)|_{x=\pm 1} = f'(x)|_{x=\pm 1}$. The derivatives of $f(x)$ and $\mathcal{C}f(x)$ are shown at Figure 49

Example 1.6 Consider the nonconvex function $F(v)$ called a *two-well* function

$$F(v) = \min\{(v-1)^2, (v+1)^2\}.$$

F is the minimum of two convex functions (wells).

It is easy to see that the convex envelope $\mathcal{C}F$ is

$$\mathcal{C}F(v) = \begin{cases} (v+1)^2 & \text{if } \ v \leq -1, \\ 0 & \text{if } \ v \in (-1, 1), \\ (v-1)^2 & \text{if } \ v \geq 1. \end{cases}$$

The next example deals with more general case:

Example 1.7 Consider a two-well function

$$F(v) = \min\{W_1(v), W_2(v)\}, \quad W_1 = av^2, \ W_2 = b\,v^2 + 1, \tag{10}$$

where parameters are arranged as $0 < a < b$.

Compute convex envelope $\mathcal{C}F(v)$. It coincides with either the graph of the original function or with an affine function $l(v) = A\,v + B$ that touches the original graph in two points. This affine function can be found as the common tangent $l(v)$ to both convex branches (wells) of $F(v)$.

Recall that equations of the tangent line to a convex curve $g(v)$ is $l(v) - g(v_s) = g'(v(s)(v - v_s)$, where v_s is the supporting point where the tangent touches the graph of $g(v)$.

Let v_1 and v_2 be the supporting points or the points where $\mathcal{C}F(v)$ touches $F(v)$. Compute the values of the common tangent $l(v)$ in the supporting points:

$$l'(v_1) = \left.\frac{dW_1}{dv}\right|_{v=v_1} = 2a\,v_1, \quad l'(v_2) = \left.\frac{dW_2}{dv}\right|_{v=v_2} = 2b\,v_2.$$

. where the supporting points v_1 and v_2 belong to the corresponding wells. The equation $l'(v_1) = l'(v_2)$ gives one relation between v_1 and v_2

$$av_1 = bv_2 \tag{11}$$

From (10), we write tangent lines to each well:

$$\begin{cases} l(v) = a\,v_1^2 + 2a\,v_1(v - v_1), & v_1 \in W_1 \\ l(v) = (b\,v_2^2 + 1) + 2b\,v_2(v - v_2), & v_2 \in W_2 \end{cases} \tag{12}$$

Setting $v = 0$, we obtain the the second relation:

$$a\,v_1^2 = b\,v_2^2 - 1 \tag{13}$$

Figure 3: Convex set, convex function on a convex set

and solve (11), (13) for the coordinates of the supporting points v_1 and v_2:

$$v_1 = \sqrt{\frac{b}{a(a-b)}}, \quad v_2 = \sqrt{\frac{a}{b(a-b)}}, \tag{14}$$

Using (12) and (14) we compute linear component of the envelope and the convex envelope itself:

$$\mathcal{C}F(v) = \begin{cases} av^2 & \text{if } \ |v| < v_1, \\ 2v\sqrt{\frac{ab}{a-b}} - \frac{b}{a-b} & \text{if } \ v \in [v_1, \ v_2], \\ 1 + bv^2 & \text{if } \ |v| < v_2 \end{cases} \tag{15}$$

$\mathcal{C}F$ linearly depends on v in the interval of non-convexity of F and coincides with F outside of this interval.

2 Convexity: Vector function

2.1 Convex function of vector argument

Convex set, convex hull A domain Ω in R^n is called convex if for any points x_1 and x_2 in Ω and for any t in the interval $[0, 1]$, all points $x = (1-t)\,x_1 + t\,x_2$ belong to Ω. In other words, any point of the line segment belong to Ω if its ends x_1 and x_2 are in Ω.

The convex hull or convex envelope $\mathcal{C}\Omega$ of a nonconvex set Ω is a linear space is the smallest convex set that contains Ω. It can be also defined as the set of all convex combinations $z$of points $x \in \Omega$.

$$z(x) = \{x : \ x = \sum_i (m_i x_i), \quad \forall x_i \in \Omega \quad \sum_i m_i = 1, m_i \geq 0\}$$

Particularly, the convex envelope of a set of any n points $a_1, \ldots a_n$ in R^n is a polygon

$$P(x) = \{x : \ x = \sum_{i=1}^{n} m_i\, a_i, \quad \sum_i m_i = 1, m_i \geq 0\}$$

stretched at these points. Parameters m_i with the stated properties are called the barycentric coordinates of x in the polygone P.

Convex function Consider a real-valued continuous function $f(x)$, where $x \in R^n$ belongs to a convex set Ω. Function f is called convex if the inequality (1) holds, in which x, v_1, v_2 are now n-vectors not scalars.

Another equivalent geometrical definition of convexity is: $f(x)$ is convex, if the $n+1$-dimensional set (x, z) where $x \in \Omega$ and $z \geq f(x)$ of the points above its graph $y \geq f(x)$ is convex.

Convexity in a point; Jensen inequality As in the scalar case, the function f is convex in a point x if

$$f(x) \le \sum_{i=1}^{n+1} m_i f(x+v_i) \quad \forall m_i,\ v_i\ i=1,\dots n+1, \quad \text{such that} \tag{16}$$

$$m_i > 0, \quad x+v_i \in \Omega, \quad \sum_{i=1}^{n} m_i = 1, \quad \sum_{i=1}^{n+1} m_i\, v_i = 0 \tag{17}$$

Derivatives. Hessian Convex differentiable functions satisfy inequality

$$f(y) \ge f(x) + (y-x)^? \nabla f(x) \quad \forall x,y \in \Omega \tag{18}$$

Second derivatives of a twice differentiable functions is characterized by the Hessian $H(f)$ which is a symmetric $n \times n$ matrix of the second derivatives with entries

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad i,j = 1,\dots n$$

If $f(x)$ is convex, its Hessian is non-negatively defined,

$$z^T H z \ge 0, \quad \forall z \in R^n, \quad |z| \ne 0$$

If f is strictly convex, its Hessian is positively defined.

Gradient of a convex function is monotone From (18), one can deduct that monotonicity of a derivative of a convex function, that is an analog of monotonicity of the derivative of a convex function of a scalar argument. We rewrite inequality (18) for the pair y,x instead of x,y:

$$f(x) \ge f(y) + \nabla f(y)(x-y) \quad \forall x,y \in \Omega$$

and subtract it from (18), obtaining

$$(y-x)^T(\nabla f(y) - \nabla F(x)) \ge 0$$

for all $x,y \in \Omega$. The last inequality is called the monotonicity of a vector-valued function. Monotonicity means that projection of the difference of gradients in any two points to the vector of difference between these points is non-negative. If $f(x)$ is convex, $\nabla f(x)$ is monotone.

Comment is not clear

2.2 Convex envelope. Vector case

Convex envelope $\mathcal{C}f(x), x \in R^n$ satisfies the equation

$$\mathcal{C}f(x) = \min_{\rho_1,\dots\rho_{n+1}} \sum_{i=1}^{n+1} m_i f(\rho_i), \tag{19}$$

$$x = \sum_{i=1}^{n+1} m_i \rho_i, \quad \sum_{i=1}^{n+1} m_i = 1, \quad m_i \ge 0 \tag{20}$$

Figure 4: Supporting points

Figure 5: barycentric coordinates

that is similar to the scalar case.

Definition 2.1 The *convex envelope* $\mathcal{C}F$ is a solution to the following minimal problem:

$$\mathcal{C}F(x_0) = \inf_{v:v+x_0 \in C} \frac{1}{l} \int_C F(x_0 + v(x))\, dx \quad \forall\ v : \int_C v(x)\, dx = 0. \tag{21}$$

This definition determines the convex envelope as the minimum of all parallel secant hyperplanes that intersect the graph of F; it is based on Jensen's inequality (**??**).

Supporting points To compute the convex envelope $\mathcal{C}F$ one can use the Carathéodory theorem (see [**?**, **?**]). It states that the argument $v(x) = [v_1(x), \ldots, v_n(x)]$ that minimizes the right-hand side of (21) takes no more than $n+1$ different values. This theorem refers to the obvious geometrical fact that the convex envelope consists of the supporting hyperplanes to the graph $F(v_1, \ldots, v_n)$. Each of these hyperplanes is supported by no more than $(n+1)$ arbitrary points.

The Carathéodory theorem allows us to replace the integral in the right-hand side of the definition of $\mathcal{C}F$ by the sum of $n+1$ terms; the definition (21) becomes:

$$CF(x) = \min_{m_i \in M} \min_{v_i \in v} \left(\sum_{i=1}^{n+1} m_i F(x + v_i) \right), \tag{22}$$

where

$$M = \left\{ m_i : \quad m_i \geq 0, \quad \sum_{i=1}^{n+1} m_i = 1 \right\} \tag{23}$$

and

$$v = \left\{ \ v_i : \quad \sum_{i=1}^{n+1} m_i v_i = 0 \ \right\}. \tag{24}$$

Parameters m_i are called barycentric coordinates of the convex hull stretched at the vertices $x + v_i$.

The convex envelope $\mathcal{C}F(x)$ of a function $F(x)$ at a point x coincides with either the function $F(x)$ or the hyperplane that touches the graph of the function F. The hyperplane remains below the graph of F except at the tangent points where they coincide.

The position of the supporting hyperplane generally varies with the point x. Fewer than $n+1$ points can support a convex envelope of F; in this case, several of the parameters m_i are zero.

Figure 6: Convex envelope: Cone and paraboloid

On the other hand, the convex envelope is the greatest convex function that does not exceed $F(x)$ in any point x [**?**]:

$$\mathcal{C}F(x) = \max \phi(x) : \ \phi(x) \le F(x) \ \forall x \quad \text{and } \phi(x) \text{ is convex.} \tag{25}$$

Example 2.1 Obviously, the convex envelope of a convex function coincides with the function itself, so all m_i but m_1 are zero in (22) and $m_1 = 1$; the parameter v_1 is zero because of the restriction (24).

The convex envelope of a "two-well" function,

$$\Phi(x) = \min \left\{ F_1(x), \ F_2(x) \right\}, \tag{26}$$

where $F_1, \ F_2$ are convex functions of x, either coincides with one of the functions $F_1, \ F_2$ or is supported by no more than two points for every x; supporting points belong to different wells. In this case, formulas (22)–(24) for the convex envelope are reduced to

$$\mathcal{C}\Phi(x) = \min_{m,v} \left\{ mF_1(x - (1-m)v) + (1-m)F_2(x + mv) \right\}. \tag{27}$$

Indeed, the convex envelope touches the graphs of the convex functions F_1 and F_2 in no more than one point. Call the coordinates of the touching points $x + v_1$ and $x + v_2$, respectively. The restrictions (24) become $m_1 v_1 + m_2 v_2 = 0, \ m_1 + m_2 = 1$. It implies the representations $v_1 = -(1-m)v$ and $v_2 = mv$.

Example 2.2 Consider the special case of the two-well function,

$$F(v_1, v_2) = \begin{cases} 0 & \text{if } \ v_1^2 + v_2^2 = 0, \\ 1 + v_1^2 + v_2^2 & \text{if } \ v_1^2 + v_2^2 \ne 0. \end{cases} \tag{28}$$

Indeed, the graph of the function $F(v_1, v_2)$ is axisymmetric in the plane $v_1, \ v_2$; therefore, the convex envelope is axisymmetric as well: $\mathcal{C}F(v_1, v_2) = f(\sqrt{v_1^2 + v_2^2})$. It is therefore enough constructing the envelope of function $F(v)$, where $v = \sqrt{v_1^2 + v_2^2}$

$$F(v) = \begin{cases} 0 & \text{if } \ v = 0, \\ 1 + v^2 & \text{if } \ v^2 \ne 0. \end{cases} \tag{29}$$

The convex envelope $\mathcal{C}F(v)$ is supported by the point $v_a = 0$ and by a point v_b that (i) belongs to the parabola $f(v) = 1 + v_b^2$ and (ii) is such that the tangent line to the parabola at the point v_b passes through the origin. The equation of the tangent line in the plane v, y is $y - f(v_b) = f'(b)(v - v_b)$. Setting $y = v = 0$ due to (ii), we find $f(v_b) = f'(v_b)v_b$ or $1 + v_b^2 = 2v_b^2$ and $v_b = 1$. The values of F are: $F(v_1) = 0, \ F(V_b) = 2$, and the envelope is $\mathcal{C}F(v) = 2v$, if $0 \le v \le 1$. Coming back to original notations we find the supporting circumferences of $F(v_1, v_2)$:

$$A : \ (v_1, v_2) = (0, 0), \quad B : \ (v_1, v_2) : \ v_1^2 + v_2^2 = 1$$

Figure 7: Three-well function

and the surface of the envelope is

$$\mathcal{C}F(v_1, v_2) = \begin{cases} 2\sqrt{v_1^2 + v_2^2} & \text{if } \; v_1^2 + v_2^2 \le 1, \\ 1 + v_1^2 + v_2^2 & \text{if } \; v_1^2 + v_2^2 > 1. \end{cases} \tag{30}$$

The envelope is a cone if it does not coincide with F, $\mathcal{C}F < F$, and a paraboloid if it coincides with F, $\mathcal{C}F = F$.

Hessian of Convex Envelope We mention here property of the convex envelope that we will use later. If the convex envelope $\mathcal{C}f(x)$ does not coincide with $f(x)$ for some $x = x_0$, then $\mathcal{C}F(x_0)$ is convex, but not strongly convex. At these points the Hessian $H((f)$ is semipositive; its determinant is zero:

$$H(\mathcal{C}f(x)) \ge 0, \quad \det H(\mathcal{C}f(x)) = 0 \quad \text{if } \mathcal{C}f < f \tag{31}$$

which say that $H(\mathcal{C}f)$ is a nonnegative degenerate matrix. These relations can be used to compute $\mathcal{C}f(x)$.

For example, compute the Hessian H of the cone $F(v_1, v_2) = 2\sqrt{v_1^2 + v_2^2}$, from (30). We have

$$H = \frac{1}{(v_1^2 + v_2^2)^{\frac{3}{2}}} \begin{pmatrix} v_2^2 & -v_1 v_2 \\ -v_1 v_2 & v_1^2 \end{pmatrix}$$

and we see that $\det(H) = 0$.

2.3 Convex envelope of a three-well function

The convex envelope is a multi-face surface. The next problem demonstrates the variety of the components of its surface.

Describe convex envelope $\mathcal{C}f$ of three-well function $f(x_1, x_2)$

$$f(x_1, x_2) = \min\{\phi_1, \phi_2, \phi_3\} \tag{32}$$

$$\phi_1 = x_1^2 + x_2^2 \tag{33}$$

$$\phi_2 = x_1^2 + (x_2 - 1)^2 \tag{34}$$

$$\phi_3 = (x_1 - 1)^2 + x_2^2 \tag{35}$$

Convex functions ϕ_i are called wells.

The convex envelope is a multi-face surface that is stretched between the wells. No more than three supporting points support each component of the envelope; the convex wells contain no more than one supporting point each.

The convex envelope is a solution to the optimization problem

$$\mathcal{C}f(x) = \min_m \min_\rho \sum_{i=1}^{3} m_i \phi_i(\rho_i) \tag{36}$$

$$x = m_1\rho_1 + m_2\rho_2 + m_3\rho_3, \tag{37}$$

$$m_1 + m_2 + m_3 = 1, \quad m_i \ge 0, \quad i = 1, 2, 3. \tag{38}$$

Here, m_i are barycentric coordinates of x in the triangle with vertices at ρ_i.

Convex envelope $\mathcal{C}f$ consists of several components:

Bottom component The bottom part Ω_0 is correspond to the case when all $m_i > 0$; the minimization with respect to ρ_i gives: =

$$\rho_1 = (0,0), \quad \rho_2 = (1,0), \quad \rho_3 = (0,1)$$

The envelope is supported by three points ρ_i in three wells. Argument x belongs to a convex hull Ω_0, stretched on these points $x \in \Omega_0$,

$$\Omega_0 = \{x_1, x_2 : \ (x_1, x_1, x_2) = \sum_{i=1}^{3} \mu_i \rho_i, \quad \sum_{i=1}^{3} \mu_i = 1, \quad \mu_i \geq 0\}$$

We compute:

$$\Omega_0 = \{x_1, x_2 : \ x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}, \tag{39}$$

The values of ϕ_i are, respectively:

$$\phi_1(\rho_1) = 0, \quad \phi_2(\rho_2) = 0, \quad \phi_3(\rho_3) = 0$$

The convex envelope in Ω_0 is

$$\mathcal{C}f(x_1, x_2) = \sum_{i=1}^{3} \mu_i \phi_i(\rho_i) = 0 \quad \text{if} \quad (x_1, x_2) \in \Omega_0, \tag{40}$$

The coordinates of a point in the convex hull are

$$x_1 = \mu_2, \quad x_2 = \mu_3, \quad 0 \leq \mu_2 + \mu_3 \leq 1, \quad \mu_2 \geq 0, \quad \mu_3 \geq 0$$

Notice that supporting points do not vary with $x \in \Omega_0$, only the barycentric coefficients μ_i ($\mu_3 = 1 - \mu_1 - \mu_2$) vary.

Side components First side component of the boundary corresponds to the case when $m_3 = 0$. This component is supported by by two points at two two convex wells ϕ_1 and ϕ_2. The domain Ω_1 that support this case, is

$$\Omega_1 = \{x_1, x_2 : \ (x_1, x_2) = \mu_1(0, x_2) + \mu_2(1, x_2), \quad m_1 + \mu_2 = 1, \quad \mu_i \geq 0$$

it is a strip:

$$\Omega_1 = \{x_1, x_2 : \ x_1 \in [0,1], \quad x_2 \in [-\infty, 0], \tag{41}$$

The supporting points are

$$\rho_1 = (0, x_2), \quad \rho_2 = (1, x_2)$$

We compute

$$\phi_1(0, x_2) = x_2^2, \quad \phi_1(1, x_2) = x_2^2,$$

Figure 8: Contourplot of the convex envelope

The convex envelope in the region Ω_2 is

$$\mathcal{C}f(x_1, x_2) = \mu_1\phi_1(0, x_2) + \mu_2\phi_2(1, x_2) = x_2^2, \quad (x_1, x_2) \in \Omega_1, \tag{42}$$

(Here, the coordinate x_1 is $x_1 = \mu_2$ and $x_1 \in (0, 1)$. This part lies between two convex wells ϕ_1 and ϕ_2 and consists of moving parallel intervals supported by two points at these wells. This type of surface is called a ruled surface, that is a surface that can be swept out by moving a line in space. A variation of position $x \in \Omega_2$ along the direction x_1 results in the variation of $\mu_1 = 1 - \mu_2$ with a fixed position of the supporting points, and a variation along the direction x_2 results in the variation of supporting points ρ_1 and ρ_2 with a fixed fraction m_2.

The second side component of the envelope correspond to $m_2 = 0$ and $m_1, m_3 > 0$. This part is similar to the previous case, it is obtained from it by interchanging indices. We have

$$\Omega_2 = \{x_1, x_2 : \ x_2 \in [0, 1], \quad x_1 \in [-\infty, 0]\} \tag{43}$$
$$\rho_1 = (x_1, 0), \quad \rho_3 = (x_1, 1)$$
$$\mathcal{C}f(x_1, x_2) = x_1^2, \quad (x_1, x_2) \in \Omega_2 \tag{44}$$

The third component correspond to $m_1 = 0$ and $m_2, m_3 > 0$. Similarly to the previous case we compute,:

$$\Omega_3 = \{x_1, x_2 : \ |x_1 - x_2| \in [0, 1], \quad x_1 + x_2 \in [1, \infty]\} \tag{45}$$
$$\rho_2 = (x_1, 0), \quad \rho_3 = (x_1, 1)$$
$$\mathcal{C}f(x_1, x_2) = (x_1 + x_2)^2, \quad (x_1, x_2) \in \Omega_3 \tag{46}$$

Regions of convexity The remaining three regions correspond to the case when one of coordinates m_i equals to one, and the other two are equal to zero. In these cases, the convex envelope coincides with the function itself, $f(x)$ is convex in these regions.

We compute

$$\mathcal{C}f = \phi_1, \quad \text{in } \Omega_4 = \{(x_1, x_2) : \ x_1 \le 0, x_2 \le 0 \tag{47}$$
$$\mathcal{C}f = \phi_2, \quad \text{in } \Omega_5 = \{(x_1, x_2) : x_2 > 1, \ 1 \ge x_2 - x_1 \tag{48}$$
$$\mathcal{C}f = \phi_3, \quad \text{in } \Omega_6 = \{(x_1, x_2) : x_1 > 1, \ 1 \ge x_1 - x_2 \tag{49}$$

In Figure 8 the contour plot of the obtained convex envelope is shown.