Constrainted variational problems

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Contents

1 Isoperimetric problem

1.1 Stationarity conditions

Isoperimetric problem of the calculus of variations asks for minimum of one integral functional subject to condition that another integral functional is fixed. A classical example is the problem of the domain of maximal area with fixed perimeter; this problem gave the name "isoperimetric" to these problems. The isoperimetric problem is formulated as follows

$$\min_u \int_a^b F(x,u,u')dx \quad \text{subject to} \quad \int_a^b G(x,u,u')dx = 0 \tag{1}$$

Applying the same procedure as in the finite-dimensional problem, we reformulate the problem using Lagrange multiplier λ:

$$\min_u \int_a^b \left[F(x,u,u') + \lambda G(x,u,u')\right] dx \tag{2}$$

To justify the approach, we consider the finite-dimensional analog of the problem

$$\min_{u_i} \sum_{i=1}^N F(x_i, u_i, \mathrm{Diff}(u_i))(x_i - x_{i-1}) \quad \text{subject to} \quad \sum_{i=1}^N G(x, u_i, \mathrm{Diff}(u_i))(x_i - x_{i-1}) = 0$$

where

$$u_i = u(x_i), \quad \mathrm{Diff}(u_i) = \frac{u_i - u_{i-1}}{x_i - x_{i-1}}, \quad |x_i - x_{i-1}| \to 0, \text{ when } N \to \infty$$

The Lagrange method is applied to the last problem which becomes

$$\min_{u_i} \sum_{i=1}^N \left[F(x_i, u_i, \mathrm{Diff}(u_i)) + \lambda G(x_i, u_i, \mathrm{Diff}(u_i))\right].$$

Passing to the limit when $N \to \infty$ we arrive at (2).

The procedure of solution is as follows: First, we solve Euler equation for the problem (2)

$$\frac{d}{dx}\frac{\partial}{\partial u'}(F + \lambda G) - \frac{\partial}{\partial u}(F + \lambda G) = 0.$$

keeping λ undefined, and arrive at minimizer $\hat{u}(x,\lambda)$ which depends on parameter λ. This parameter is defined from equation

$$\int_a^b G(x, u(\hat{x,\lambda}), u(\hat{x,\lambda})')dx = 0$$

Remark 1.1 The method assumes that the constraint is consistent with the variation: The variation must be performed in a class of functions u that satisfy the constraint. Parameter λ has the meaning of the cost for violation of the constraint.

Of course, it is assumed that the constraint can be satisfied for all varied functions that are close to the optimal one. For example, the method is not applicable to the constraint

$$\int_a^b u^2 dx \le 0$$

because this constraint allows for only one function $u = 0$ and will be violated at any varied trajectory.

Generalization The isoperimetric problem is naturally generalized to several constraints and to inequality constraints, as is is done for the finite-dimensional optimization problem. Namely, consider the minimization problem

$$\min_u \int_a^b F(x,u,u')dx \quad \text{subject to} \quad \int_a^b G(x,u,u')dx = 0 \tag{3}$$

subject to integral constraints

$$\int_a^b G_i(x,u,u')dx = 0 \quad i = 1,\dots,k \tag{4}$$

$$\int_a^b H_j(x,u,u')dx < 0 \quad j = 1,\dots,m. \tag{5}$$

The extended Lagrangian is

$$L(u,u',\lambda,\mu) = F(x,u,u')dx + \sum_{i=1}^{k} \lambda_i G_i(x,u,u') + \sum_{j=1}^{m} \mu_j H_j(x,u,u') \tag{6}$$

where $\lambda = (\lambda_1 \dots \lambda_k)$ and $\mu = (\mu_1 \dots \mu_m)$ are vectors of Lagrange multipliers λ_i and μ_j, respectively.

The stationarity conditions

$$\left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u'}\right) L(u,u',\lambda,\mu) = 0, \tag{7}$$

$$\mu_j \int_a^b H_j(x,u,u')dx = 0, \quad \mu_j \geq 0, \quad j = 1,\dots,m \tag{8}$$

allow to determine the solution that depends on Lagrange multipliers, $u(x,\lambda,\mu)$ and satisfies constraints (4), (5), (8).

1.2 Dido problem

The first isoperimetric problem has been solved by legendary wise princess Dido, founder and queen of Carthage, the mighty rival of Rome:

> The problem was as follows: What is the maximum area of land that can be encircled by a rope of given length?
>
> Solution: The circle or (if the land is on the sea shore) the semicircle.

The problem is described in a passage from Virgil's Aeneid:

> *"The Kingdom you see is Carthage, the Tyrians, the town of Agenor;*
> *But the country around is Libya, no folk to meet in war.*
> *Dido, who left the city of Tyre to escape her brother,*
> *Rules here–a long a labyrinthine tale of wrong*

Is hers, but I will touch on its salient points in order....
Dido, in great disquiet, organized her friends for escape.
They met together, all those who harshly hated the tyrant
Or keenly feared him: they seized some ships which chanced to be ready...
They came to this spot, where to-day you can behold the mighty
Battlements and the rising citadel of New Carthage,
And purchased a site, which was named 'Bull's Hide' after the bargain
By which they should get as much land as they could enclose with a bull's hide."

In modern notations, the problem is as follows: Consider a bounded domain Ω in a plane and its smooth boundary γ. Introduce the parametric representation of the closed contour γ on a plane

$$\gamma: \ \{(x(s), y(s)), \quad s \in [s_0, s_1], \quad x(s_0) = x(s_1), \quad y(s_0) = y(s_1)\}$$

The length L and the area A of the contour are expressed through the coordinates of the contour as

$$L = \int_{s_0}^{s_1} \sqrt{x'^2 + y'^2}\, ds \tag{9}$$

$$A = \int_\Omega dx\, dy = \frac{1}{2}\int_{s_0}^{s_1} (xy' - x'y)\, ds \tag{10}$$

where $x' = \frac{dx}{ds}$, $y' = \frac{dy}{ds}$.

Remark 1.2 Formula (10) is obtained by applying Green's formula (Divergence theorem):

$$\int_\Omega \operatorname{div} F\, dx\, dy = \oint_\gamma F \cdot \nu\, ds \tag{11}$$

where $F(x, y)$ is a vector function, ν is the normal to the boundary γ,

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad \operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}, \quad \nu = \begin{pmatrix} y' \\ -x' \end{pmatrix}$$

and the path of integration is anticlockwise. If F is set as follows: $F_1 = \frac{1}{2}x$, $F_2 = \frac{1}{2}y$, the divergence theorem gives (9).

The isoperimetric problem asks for the shape Ω of maximum of A if the length L is fixed. Accounting for the constraint by Lagrange multiplier λ, we arrive at the problem

$$I = \max_{x(s),\ y(s)} (A + \lambda L) \tag{12}$$

The Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2}(xy' - x'y) + \lambda\sqrt{x'^2 + y'^2}$$

Euler-Lagrange equations are

$$\frac{d\mathcal{L}}{ds}\frac{\partial \mathcal{L}}{\partial x'} - \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{d\,s}\left(-\frac{1}{2}y + \frac{\lambda x'}{\sqrt{x'^2 + y'^2}}\right) - \frac{1}{2}y' = 0 \tag{13}$$

$$\frac{d\mathcal{L}}{ds}\frac{\partial \mathcal{L}}{\partial xy'} - \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{d\,s}\left(\frac{1}{2}x + \frac{\lambda y'}{\sqrt{x'^2 + y'^2}}\right) + \frac{1}{2}x' = 0 \tag{14}$$

they can be integrated. We have

$$-y + \frac{2\lambda x'}{\sqrt{x'^2 + y'^2}} = C_1, \quad x + \frac{2\lambda y'}{\sqrt{x'^2 + y'^2}} = C_2 \tag{15}$$

where C_1 and C_2 are constant of integration. Rearranging terms, we write

$$\frac{\lambda}{\sqrt{x'^2 + y'^2}} = \frac{C_1 + y}{x'} = \frac{C_2 - x}{y'} \tag{16}$$

The second equality in (16) gives $(C_1 + y)y' + (x - C_2)x' = 0$. or

$$\frac{d}{ds}\left[(C_1 + y)^2 + (x - C_2)^2\right] = 0$$

Integration of this expression gives the solution:

$$(C_1 + y)^2 + (x - C_2)^2 = R^2.$$

where R^2 is the constant of integration. It shows that the optimal curve is a circle. Its radius R is found from the constraint (9): $2\pi R = L$. Constants C_1 and C_2 represent the coordinates of the center of the circle; they are arbitrary.

1.3 Catenary

The classical problem of the shape of a heavy chain (catenary, from Latin *catena* which means "chain") was considered by Euler. Assume that a heavy chain of the shape $y(x)$ hangs between points $(-a, 0)$ and $(a, 0)$ in a vertical plane. The length of the chain

$$L = \int_0^1 \sqrt{1 + (y')^2}dx$$

is fixed, the coordinates of the ends are fixed too $y(-a) = 0$ and $y(a) = 0$.

We postulate, that the equilibrium shape of the chain minimizes its potential energy W that is an integral of potential energy of the links of the chain. The potential energy dW of an infinitesimal link is $dW = g\rho(x)y(x)ds$ where g is gravitational acceleration, ρds is the the weight of an infinitesimal element, ρ is the density, $ds = \sqrt{1 + (y')^2}dx$ is the length of of the chain' element.

The whole energy of the chain is

$$W = \int_{-a} ag\rho y\, ds = g\rho \int_0^1 y\sqrt{1 + (y')^2}dx$$

Normalizing, we put $g\rho = 1$.

The problem becomes a constrained optimization problem: minimize the energy W when the length L is fixed. The Lagrangian $\mathcal{L}$ is $\mathcal{L} = W(y,y') + \lambda L(y,y')$ where λ is Lagrange multiplier, and the variational problem is

$$I = \min_{y(x)} \int_{-a}^{a} \mathcal{L}(y,y')dx \quad \mathcal{L}(y,y') = \int_0^1 (y+\lambda)\sqrt{1+(y')^2}dx$$

The Lagrangian is independent of x and therefore permits the first integral

$$y'\frac{\partial \mathcal{L}}{\partial y'} - \mathcal{L} = (y+\lambda)\left(\frac{(y')^2}{\sqrt{1+(y')^2}} - \sqrt{1+(y')^2}\right) = C$$

that is simplified to

$$\frac{y+\lambda}{\sqrt{1+(y')^2}} = C.$$

We solve for y'

$$\frac{dy}{dx} = \sqrt{\left(\frac{y+\lambda}{C}\right)^2 - 1},$$

and integrate

$$x = \ln\left(\lambda + y + \sqrt{\left(\frac{y+\lambda}{C}\right)^2 - 1}\right) - \ln C + x_0.$$

and find $y(x)$

$$y = -C\cosh\left(\frac{x-x_0}{C}\right) + \lambda$$

The constants x_0, C, and λ are found from the boundary conditions and the constraint. By symmetry, $x_0 = 0$, and the two remaining conditions are transformed to two equations for C and λ:

$$y(a) = y(-a) = -C\cosh\left(\frac{a}{C}\right) + \lambda = 0$$
$$\int_{-a}^{a} y(x)dx = \sinh(C\,a) + 2\lambda a = L$$

Obviously, this system has a solution only if $L < 2a$.

The equation – the catenary – defines the shape of a chain; it also gave the name to the hyperbolic cosine.

1.4 An optimal design problem

The energy density F of the bended beam equals to

$$F(w,w'') = \frac{1}{2}D(w'')^2 - q\,w$$

where $w = w(x)$ is the deflection, $q = q(x)$ is the density of the load, $D = D(x)$ is the beam's stiffness. The integral of F represents the energy stored in the beam bended under the load; it characterizes the integral stiffness of the beam: the smaller is the integral, the smaller is the norm of deflection under the given load q, the stiffer is the beam.

Differential equation of the equilibrium of the beam is the Euler equation of the variational problem

$$I = \min_{w(x)} \int_a^b F(w, w'')\, dx$$

and is (check it)

$$[Dw'']'' = q; \tag{17}$$

this equation is integrated with the the boundary conditions that specify the type of fasting of the beam at the ends.

Stiffness $D > 0$ is proportional to the width $S(x)$ of the beam, $D(x) = \kappa S(x) > 0$; the volume of the beam is proportional to the integral of $S(x)$. Consider the optimal design problem: Choose $S(x)$ to minimize I, if the total volume of the beam is given or integral of the width is given

$$\int_a^b S(x) dx = v \tag{18}$$

The Lagrangian

$$L = F(w, w'', S) + \lambda \left(S - \frac{v}{b-a} \right)$$

where λ is the Lagrange multiplyer, depends on two minimizers w and S. The Euler equation for w is (17); the Euler equation for δS is an algebraic relation because L does not depend on derivative S'. This stationarity equation

$$\frac{dL}{dS} = \frac{1}{2}(w'')^2 + \lambda = 0 \tag{19}$$

shows that $|w''|$ is constant in an optimal beam no matter what the load q is, and $\lambda < 0$.

The equation (17) becomes:

$$\operatorname{sign}(w'') S'' + \frac{q}{\kappa\sqrt{-\lambda}} = 0 \tag{20}$$

Together with (19) and boundary conditions, it allows for finding w and S.

Cantilever beam The boundary conditions are especially simple for a cantilever beam, clamped at the point $x = a$ and free at the point $x = b$. They allow to find an analytic solution to the problem. These conditions are:

$$w(a) = 0, \quad w'(a) = 0, \tag{21}$$

$$D(b)w''(b) = 0, \quad [D(b)w''(b)]' = 0 \tag{22}$$

Assume that $w(x)''$ keeps its sign everywhere (which correspond to the loading $q(x) > 0$, as we will see). Integrating (20), we find:

$$w(x) = \frac{q}{2\kappa\sqrt{-\lambda}}(x-a)^2$$

Equation (20) is a separate one; it is integrated with boundary conditions (22) that take the form $S(b) = 0, S'(b) = 0$ The solution is

$$S(x) = \frac{1}{\kappa\lambda}\int_x^b dz \int_z^b q(\psi)d\psi.$$

Finally, we use condition (18) to find λ.

Remark 1.3 This example illustrates a principle of optimal design: the variable material property (here, the width $S(x)$) of the optimally designed construction adapts itself to the loading $q(x)$ while the mechanical behavior of the construction (here the curvature w'') is is less dependent on that variable loading.

Example 1.1 Assume that $q(x) = 1$ is constant. Then

$$S(x) = \frac{q}{2\kappa\sqrt{-\lambda}}(x-b)^2.$$

Lagrange multiplier λ is determined from the constraint (18):

$$v = \frac{q}{2\kappa\sqrt{-\lambda}}\int_0^b (x-b)^2 dx = \frac{q\,b^3}{12\kappa\sqrt{-\lambda}}$$

or

$$\frac{1}{\sqrt{-\lambda}} = \frac{12vk}{qv^3}$$

and the width of an optimal beam is

$$S(x) = \frac{6}{v^2}(x-b)^2$$

Problem Find optimal width of the simple supported beam:

$$w(0) = w(b) = 0, \quad w''(0) = w''(b) = 0 \quad \text{if } q(x) = 1$$

1.5 Homogeneous functionals and Eigenvalue Problem

The next two problems are *homogeneous*: The functionals $I(u)$ stay invariant if the solution is multiplied by any real number k,

$$I(u) = I(k\,u) \quad \forall k \in R$$

Therefore, the solution $u(x)$ is defined up to a constant multiplier.

An eigenvalue problem The eigenvalue problem corresponds to the functional

$$I_1 = \min_u \frac{\int_0^1 (u')^2 dx}{\int_0^1 u^2 dx} \quad u(0) = u(1) = 0 \tag{23}$$

Because the solution is defined up to a multiplier, we can normalize it assuming that

$$\int_0^1 u^2 dx = 1 \tag{24}$$

Then the problem becomes a problem of the constrained minimum. Accounting for the normalized constraint (24) and using Lagrange multiplyer λ, we bring it to the standard form.

$$I_1 = \min_u \int_0^1 \left((u')^2 + \lambda(u^2 - 1)\right) dx \quad x(0) = x(1) = 0.$$

The Euler equation is

$$u'' - \lambda u = 0, \quad u(0) = u(1) = 0$$

This equation represents the eigenvalue problem. It has nonzero solution $u(x) = C\sin(\sqrt{-\lambda}x)$ only if λ takes special values, the eigenvalues, which are chosen so, that the solution satisfs boundary condition

$$u(1) = C\sin(\sqrt{-\lambda}) = 0$$

The eigenvalues are $\lambda_n = -(\pi n)^2$ where n is an integer. The corresponding solutions – the eigenfunctions u_n – are equal to

$$u_n(x) = C\sin(\pi n x)$$

. The constant C is determined from the normalization (24) as $C = \sqrt{2}$. However, the cost $I_1(u_n)$ of the problem is independent of C

$$I_1 = \frac{\int_0^1 (u')^2 dx}{\int_0^1 u^2 dx} = n^2\pi^2$$

The minimal cost I_1 corresponds to $n = 1$ (the first eigenvalue) and is equal to $I_1 = \pi^2$

A homogeneous problem The homogeneous problem may have only one solution defined up to a multiplier. For example, consider the problem:

$$I_2 = \min_u \frac{\int_0^1 (u')^2 dx}{\left(\int_0^1 u dx\right)^2} \quad u(0) = u(1) = 0 \tag{25}$$

The problem (25) is homogeneous, and its solution u is defined up a multiplier. As in the previous example, reformulate the problem by normalizing the solution,

$$\int_0^1 u dx = 1. \tag{26}$$

The problem (25) becomes

$$\min_u \int_0^1 \left((u')^2 + \lambda u\right) dx \quad u(0) = u(1) = 0$$

where λ is the Lagrange multiplier by the normalization constraint.

The minimizer u satisfies Euler equation

$$u'' - \frac{\lambda}{2} = 0, \quad u(0) = u(1) = 0$$

and is equal to $u(x) = \frac{\lambda}{4}x(x-1)$. The constraint (26)gives $\lambda = -24$ and the objective is

$$\int_0^1 (u')^2 dx = 36 \int_0^1 (2x-1)^2 dx = 12$$

One can check that the cost of the problem (25) is invariant to the constant of normalization or to the vallue of λ.

Remark 1.4 The two onsidered homogeneous variational problems correspond to different types of Euler equation. The equation for the problem (23) is has either infinitely many solutions for special values λ of or no solutions for other values of it. The Euler equation can point to the set of stationary solutions but cannot select a solution inside the set: this is done by straight comparison of the objective functionals.

The problem (25) leads to hon-homogeneous Euler equation that linearly depend on the constant λ of normalization. It has a unique solution if the normalization constant is fixed.

2 General form of a variational functional

2.1 Reduction to isoperimetric problem

Lagrange method allows for reformulation of an extremal problem in a general form as a simplest variational problem. The minimizing functional can be the product, ratio, superposition of other differentiable function of integrals of the minimizer and its derivative. Consider the problem

$$J = \min_u \Phi(I_1, \dots, I_n) \tag{27}$$

where

$$I_k(u) = \int_a^b F_k(x, u, u') dx \quad k = 1, \dots n \tag{28}$$

and Φ is a continuously differentiable function. Using Lagrange multipliers λ_1, λ_n, we transform the problem (27) to the form

$$J = \min_u \min_{I_1,\dots,I_n} \max_{\lambda_1,\dots\lambda_n} \left\{ \Phi + \sum_{k=1}^n \lambda_k \left(I_k - \int_a^b F_k(x,u,u')dx \right) \right\}. \tag{29}$$

The stationarity conditions for (29) consist of n algebraic equations

$$\frac{\partial \Phi}{\partial I_k} + \lambda_i = 0 \tag{30}$$

and the differential equation – the Euler equation

$$S(\Psi, u) = 0$$

$$\left(\text{recall that } S(\Psi, u) = \frac{d}{dx}\frac{\partial \Psi}{\partial u'} - \frac{\partial \Psi}{\partial u} \right)$$

for the function

$$\Psi(u) = \sum_{k=1}^n \lambda_k F_k(x,u,u')$$

Together with the definitions (28) of I_k, this system enables us to determine the real parameters I_k and λ_k and the function $u(x)$. The Lagrange multipliers can be excluded from the previous expression using (30), then the remaining stationary condition becomes an integro-differential equation

$$S(\bar{\Psi}, u) = 0, \quad \bar{\Psi}(I_k, u) = \sum_{k=1}^n \frac{\partial \Phi}{\partial I_k} F_k(x,u,u') \tag{31}$$

Next examples illustrate the approach.

2.1.1 Product of integrals

Consider the problem

$$\min_u J(u), \quad J(u) = \left(\int_a^b \phi(x,u,u')dx \right) \left(\int_a^b \psi(x,u,u')dx \right).$$

We rewrite the minimizing quantity as

$$J(u) = I_1(u) I_2(u), \quad I_1(u) = \int_a^b \phi(x,u,u')dx, \quad I_2(u) = \int_a^b \psi(x,u,u')dx,$$

apply stationary condition (31), and obtain the condition

$$\delta J(u) = I_1 \delta I_2 + I_2 \delta I_1 = I_2(u) S(\phi(u), u) + I_1(u) S(\psi(u), u) = 0. \tag{32}$$

or

$$\left(\int_a^b \phi(x,u,u')dx\right) S(\psi(u),u) + \left(\int_a^b \psi(x,u,u')dx\right) S(\phi(u),u) = 0$$

(Recall, that $S(\phi(u),u) = \frac{\partial \phi}{\partial u} - \frac{d}{dx}\frac{\partial \psi}{\partial u'}$).

The stationary equation is nonlocal: Solution u at each point depends on its first and second derivatives and integrals of $\phi(x,u,u')$ and $\phi(x,u,u')$ over the whole interval $[a,b]$.

Example 2.1 Solve the problem

$$\min_u \left(\int_0^1 (u')^2 dx\right)\left(\int_0^1 (u+1)dx\right) \quad u(0)=0, \quad u(1)=a$$

We denote

$$I_1 = \frac{1}{2}\int_0^1 (u')^2 dx, \quad I_2 = \int_0^1 (u+1)dx$$

and compute the Euler equation using (32)

$$I_2 u'' - I_1 = 0, \quad u(0)=0, \quad u(1)=a.$$

or (assuming that $I_2 \neq 0$)

$$u'' - R = 0, \quad u(0)=0, \quad u(1)=a, \quad R = \frac{I_1}{I_2}$$

The integration gives

$$u(x) = \frac{1}{2}Rx^2 + \left(a - \frac{1}{2}R\right)x,$$

We obtain the solution that depends on R – the ratio of the integrals of two function of this solution. To find R, we substitute the expression for $u = u(R)$ into right-hand sides of I_1 and I_2,

$$I_1 = \frac{R^2}{12} + a^2, \quad I_2 = -\frac{R}{12} + \frac{1}{2}a + 1$$

compute the ratio, $\frac{I_1}{I_2} = R$ and obtain the equation for R,

$$R = \frac{R^2 + 12a^2}{R + 6a + 12}$$

Solving it, we find $R = \frac{1}{2}(3a + 6 \pm \sqrt{36 + 36a - 15a^2})$.

2.1.2 The ratio of integrals

Consider the problem

$$\min_u J(u), \quad J(u) = \frac{\int_a^b \phi(x, u, u')dx}{\int_a^b \psi(x, u, u')dx}.$$

We rewrite it as

$$J = \frac{I_1}{I_2}, \quad I_1(u) = \int_a^b \phi(x, u, u')dx, \quad I_2(u) = \int_a^b \psi(x, u, u')dx, \qquad (33)$$

apply stationary condition (31), and obtain the condition

$$\frac{1}{I_2(u)} S(\phi(u), u) - \frac{I_1(u)}{I_2^2(u)} S(\psi(u), u) = 0.$$

Multiplying this equality by I_2 and using definition (33) of the goal functional, we bring the previous expression to the form

$$S(\phi, u) - J\, S(\psi, u) = S(\phi - J\psi, u) = 0$$

Observe that the stationarity condition depends on the cost J of the problem.

Example 2.2 Solve the problem

$$\min_u J(u), \quad J = \frac{\int_0^1 (u')^2 dx}{\int_0^1 (u-1)dx} \quad u(0) = 0, \quad u(1) = a$$

We compute the Euler equation

$$u'' - J = 0, \quad u(0) = 0, \quad u(1) = a.$$

where

$$R = \frac{I_1}{I_2}, \quad I_1 = \int_0^1 (u')^2 dx, \quad I_2 = \int_0^1 (u-1)dx \qquad (34)$$

The integration gives

$$u(x) = \frac{1}{2}Rx^2 + \left(a - \frac{1}{2}R\right)x,$$

We obtain the solution $u(R)$ that depends on the ratio R of the integrals of two function of this solution. To find R, we substitute the expression for $u = u(R)$ into right-hand sides of I_1 and I_2 in (34) and compute:

$$I_1 = \frac{R^2}{12} + a^2, \quad I_2 = -\frac{R}{12} + \frac{1}{2}a - 1$$

Recalling that $\frac{I_1}{I_2} = R$ we obtain the equation for R,

$$R = -\frac{R^2 + 12a^2}{R - 6a + 12}$$

Solving this quadratic equation, we choose the smaller root $R = \frac{1}{2}(3a - 6 - \sqrt{36 - 36a - 15a^2})$.

The examples will be given in the next section.

2.1.3 Superposition of integrals

Consider the problem

$$\min_u \int_a^b R\left(x, u, u', \int_a^b \phi(t, u, u')dt\right) dx$$

We introduce a new variable I

$$I = \int_a^b \phi(t, u, u')dt$$

and reformulate the problem as

$$\min_u \int_a^b \left[R(x, u, u', I) + \lambda \left(\phi(x, u, u') - \frac{I}{b-a}\right)\right] dx$$

where λ is the Lagrange multiplier. The stationarity conditions are:

$$S((R + \lambda\phi), u) = 0, \quad \frac{\partial R}{\partial I} - \frac{1}{b-a} = 0.$$

and the above definition of I.

Example 2.3 (Integral term in the Lagrangian) Consider the following extremal problem posed in "physical terms": Find the function $u(x)$ on the interval $[0, 1]$ that is has prescribed values at its ends,

$$u(0) = 1, \quad u(1) = 0, \tag{35}$$

and minimizes a weighted sum of two terms: the L_2-norm of the derivative u'

$$P = \int_0^1 u'^2 dx,$$

and

$$Q = \int_0^1 (u - a)^2\, dx, \quad a = \int_0^1 u(t)dt. \tag{36}$$

The last requirement says that $u(x)$ stays close to its averaged over the interval $[0, x]$ value a. We combine the two above constraints on $u(x)$ into one Lagrangian F equal to the weighted sum of them:

$$F = P + \alpha Q = (u')^2 + \alpha \left(u - \int_0^1 u(t)dt\right)^2$$

where $\alpha \geq 0$ is a weight coefficient that show a relative importance of the two criteria. Function $u(x)$ is a solution to the extremal problem

$$\min_{u(x),\ u(0)=1, u(1)=0} \int_0^1 F\left(u, u', \int_0^1 u(t)dt\right) dx \tag{37}$$

We end up with the variational problem with the Lagrangian that depends on the minimizer u, its derivative and its integral.

Remark 2.1 Formulating the problem, we could include boundary conditions into a minimized part of the functional instead of postulating them; in this case the problem would be

$$\min_{u(x)} \left\{ \int_0^1 F\left(u, u', \int_0^1 u(t)dt\right) dx + \beta_1 (u(0)-1)^2 + \beta_2 u(1)^2 \right\}$$

where $\beta_1 \geq 0$ and $\beta_2 \geq 0$ are the additional weight coefficients.

We bring the problem (37) to the form of the standard variational problem, accounting for the equality (36) with the Lagrange multiplier λ; the objective functional becomes

$$J = \int_0^1 \left(u'^2 + \alpha(u-a)^2\right) dx + \lambda \left(a - \int_0^1 u\, dx\right)$$

or

$$J = \int_0^1 \left(u'^2 + \alpha(u-a)^2 + \lambda(a-u)\right) dx$$

The parameter a and the function $u(x)$ are the unknowns. The stationary condition with respect to a is

$$\frac{\partial J}{\partial a} = \int_0^1 \left(-2\alpha(u-a) + \lambda\right) dx = 2\alpha a + \lambda - 2 \underbrace{\int_0^1 u\, dx}_{=a} = 0,$$

it allows for linking a and λ,

$$\lambda = 2(1-\alpha)a.$$

The stationary condition with respect to $u(x)$ (Euler equation) is

$$2u'' - 2\alpha(u-a) + \lambda = 0$$

We exclude λ using the obtained expression for λ, and obtain

$$2u'' - 2\alpha u + a = 0 \tag{38}$$

The integro-differential system (36) and (38) with the boundary conditions (35) determines the minimizer.

To solve the system, we first solve (38) and (35) treating a as a parameter,

$$u(x) = \frac{a}{2\alpha} + A \sinh(\sqrt{\alpha}x) + B \cosh(\sqrt{\alpha}x)$$

where

$$A = \left(\frac{a}{2\alpha} - 1\right) \frac{\cosh(\sqrt{\alpha})}{\sinh(\sqrt{\alpha})}, \quad B = 1 - \frac{a}{2\alpha},$$

and substitute this solution into (36) obtaining the linear equation for the remaining unknown a. We have

$$u(x) = c_1(x)a + c_2(x)$$

where

$$c_1(x) = \frac{1}{2\alpha}\left(1 + \frac{\cosh(\sqrt{\alpha})}{\sinh(\sqrt{\alpha})}\sinh(\sqrt{\alpha}x) - \cosh(\sqrt{\alpha}x)\right)$$

and

$$c_2(x) = \left(\cosh(\sqrt{\alpha}x) - \frac{\cosh(\sqrt{\alpha})}{\sinh(\sqrt{\alpha})}\sinh(\sqrt{\alpha}x)\right)$$

and (36) becomes

$$a = a\int_0^1 c_1(x)dx + \int_0^1 c_2(x)dx$$

which implies

$$a = \frac{\int_0^1 c_2(x)dx}{1 - \int_0^1 c_1(x)dx}$$

The general procedure is similar: We always can rewrite a minimization problem in the standard form adding new variables (as the parameter c in the previous examples) and corresponding Lagrange multipliers.

Inequality in the isoperimetric condition Often, the isoperimetric constraint is given in the form of an inequality

$$\min_u \int_a^b F(x, u, u')dx \quad \text{subject to} \quad \int_a^b G(x, u, u')dx \geq 0 \tag{39}$$

In this case, the additional condition $\lambda \geq 0$ is added to the Euler-Lagrange equations (2).

Remark 2.2 Sometimes, the replacement of an equality constraint with the corresponding inequality can help to determine the sign of the Lagrange multiplier. For example, consider the Dido problem, and replace the condition that the perimeter is fixed with the condition that the perimeter is smaller than or equal to a constant. Obviously, the maximal area corresponds to the maximal allowed perimeter and the constraint is always active. On the other hand, the problem with the inequality constraint requires positivity of the Lagrange multiplier; so we conclude that the multiplier is positive in both the modified and original problem.

Homogeneous with a power functionals To complete the considerations, consider a larger class of homogeneous with a power p functionals, $I(qu) = q^p I(u)$ where $q > 0$ is an arbitrary positive number. For example function $I(x) = ax^4$ is homogeneous with the power four, because $I(qx) = aq^4x^4 = q^4 I(x)$. Here, $p \neq 1$ is a real number. for all u. For example, the functional can be equal to

$$J_3(u) = \frac{\int_0^1 (u')^2 dx}{\left|\int_0^1 u dx\right|^p}, \quad u(0) = u(1) = 0, \quad u \not\equiv 0 \tag{40}$$

which implies that it is homogeneous with the power $2-p$, because $J_3(qu) = q^{2-p} J_3(u)$.

The minimization of such functionals leads to a trivial result: Either $\inf_u J_3 = 0$ or $\inf_u J_3 = -\infty$, because the positive factor q^p can be made arbitrarily large or small.

More exactly, if there exist u_0 such that $I(u_0) \leq 0$, than $\inf_u J_3 = -\infty$; the minimizing sequence consists of the terms $q_k u_0$ where the multipliers q_k are chosen so that $\lim q_k^p = \infty$.

If $I(u_0) \geq 0$ for all u_0, than $\inf_u J_3 = 0$; the minimizing sequence again consists of the terms $q_k u_0$ where the multipliers q_k are chosen so that $\lim q_k^p = 0$.

Remark 2.3 In the both cases, the minimizer itself does not exist but the minimizing sequence can be built. These problems are examples of variational problems without classical solution that satisfies Euler equation. Formally, the solution of problem (40) does not exist because the class of minimizers is open: It does not include $u \equiv 0$ and $u \equiv \infty$ one of which is the minimizer. We investigate the problems without classical solutions in Chapter ??.

2.2 Constraints in boundary conditions

Constraints on the boundary, fixed interval Consider a variational problem (in standard notations) for a vector minimizer u. If there are no constrains imposed on the end of the trajectory, the solution to the problem satisfies n natural boundary conditions

$$\delta u(b) \cdot \left.\frac{\partial F}{\partial u'}\right|_{x=b} = 0$$

(For definiteness, we consider here conditions on the right end, the others are clearly identical).

The vector minimizer of a variational problem may have some additional constraints posed at the end point of the optimal trajectory. Denote the boundary value of $u_i(b)$ by v_i The constraints are

$$\phi_i(v_1, \ldots v_n) = 0 \quad i = 1, \ldots, k; \quad k \leq n$$

or in vector form,

$$\Phi(b, v) = 0,$$

where Φ is the corresponding vector function. The minimizer satisfies these conditions and $n-k$ supplementary natural conditions that are derived from the minimization requirement. Here we derive these supplementary boundary conditions for the minimizer.

Let us add the constrains with a vector Lagrange multiplier $\lambda = (\lambda_1, \ldots . \lambda_k)$ to the problem. The variation of $v = u(b)$ gives the conditions

$$\delta v \cdot \left[\left.\frac{\partial F}{\partial u'}\right|_{x=b, u=v} + \frac{\partial \Phi}{\partial v} \lambda \right] = 0$$

The vector in the square brackets must be zero because of arbitrariness of $\delta u(b)$.

Next, we may exclude λ from the last equation (see the previous section **??**):

$$\lambda = -\left[\left(\frac{\partial \Phi}{\partial u}\right)^T \frac{\partial \Phi}{\partial u}\right]^{-1} \frac{\partial F}{\partial u'}\bigg|_{x=b,u=v} \tag{41}$$

and obtain the conditions

$$\left(I - \frac{\partial \Phi}{\partial u}^T \left[\left(\frac{\partial \Phi}{\partial u}\right)^T \frac{\partial \Phi}{\partial u}\right]^{-1} \frac{\partial \Phi}{\partial u}\right) \frac{\partial F}{\partial u'}\bigg|_{x=b,u=v} = 0 \tag{42}$$

The rank of the matrix in the parenthesis is equal to $n-k$. Together with k constrains, these conditions are the natural conditions for the variational problem.

2.2.1 Example

$$\min_{u_1,u_2} \int_a^b (u_1'^2 + u_2'^2 + u_3')dx, \quad u_1(b) + u_2(b) = 1,\ u_1(b) - u_3(b) = 1,$$

We compute

$$\frac{\partial F}{\partial u'} = \begin{pmatrix} 2u_1 \\ 2u_2 \\ 1 \end{pmatrix}, \quad \frac{\partial \Phi}{\partial u} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(please continue..)

Free boundary with constraints Consider a general case when the constraints $\Phi(x,u) = 0$ are posed on the solution at the end point. Variation of these constrains results in the condition:

$$\delta\Phi(x,u)|_{x=b} = \frac{\partial \Phi}{\partial u}\delta u + \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u}u'\right)\delta x$$

Adding the constraints to the problem with Lagrange multiplier λ, performing variation, and collecting terms proportional to δx, we obtain the condition at the unknown end point $x = b$

$$F(x,u,u') - \frac{\partial F}{\partial u'}u' + \lambda^T \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u}u'\right) = 0$$

where λ is defined in (41). Together with $n-k$ conditions (42) and k constraints, they provide $n+1$ equations for the unknowns $u_1(b), \ldots, u_n(b), b$.

3 Pointwise Constraints

3.1 Stationarity conditions

Some variational problems deal with a vector minimizer that is subject to constraints in every point of the trajectory. For example, the problem of geodesics asks for the shortest path between two points on a surface, if the path lies on the surface. These problems are addressed by introducing a Lagrange function instead of Lagrange multipliers.

Consider a variational problem for a vector-valued minimizer $u = u_1, \dots u_n$.

$$\min_u \int_a^b F(x, u, u')dx$$

Assume that the minimizer satisfies a certain algebraic constraint in each point of any admissible trajectory,

$$G(x, u) = 0, \quad \forall x \in (a, b) \tag{43}$$

We arrive at the pointwise constrained variational problem

$$\min_u \int_a^b F(x, u, u')dx \quad \text{subject to } G(x, u) = 0, \quad \forall x \in (a, b) \tag{44}$$

As in the isoperimetric problem, we use the Lagrange multipliers method to account for the constraint. This time, the constraint must be enforced in every point of the trajectory, therefore the Lagrange multiplier becomes a function of x. To prove the method, it is enough to pass to the finite-dimensional problem; after discretization, the constraint is replaced by the set of equations

$$G(x, u) = 0 \;\Rightarrow\; G(x_i, u_i) = 0, \quad i = 1, \dots N.$$

that requires that the constraint (43) is enforced at the points x_i. Each of this constraints, multiplied by its own Lagrange multiplier $\mu_1, \dots \mu_N$, must be added to the functional. The set of these multipliers converges to a function $\mu(x)$ when $N \to \infty$. The variational problem becomes

$$\min_u \int_a^b \left[F(x, u, u') + \mu(x)G(x, u)\right] dx \tag{45}$$

The necessary conditions consist of constraints (43) and Euler equation:

$$G(x, u, u') = 0 \tag{46}$$

$$-\frac{d}{dx}\frac{\partial F}{\partial u'} + \frac{\partial F}{\partial u} + \mu \frac{\partial G}{\partial u} = 0, \tag{47}$$

that define functions $u(x)$ and $\mu(x)$.

Elimination of $\mu(x)$ In scalar form, equation (47) is written are the system of differential equations:

$$\frac{\partial F}{\partial u_k} - \frac{d}{d\,x}\frac{\partial F}{\partial u_k'} + \mu\frac{\partial G}{\partial u_k} = 0, \quad k = 1\ldots,n.$$

We can express $\mu = \mu(x)$ from each equation of the system

$$\mu(x) = \left(\frac{\partial F}{\partial u_i} - \frac{d}{d\,x}\frac{\partial F}{\partial u_i'}\right)\left(\frac{\partial G}{\partial u_i}\right)^{-1}, \quad i- = 1,\ldots,n$$

and obtain $n-1$ equations

$$\left(\frac{\partial F}{\partial u_1} - \frac{d}{d\,x}\frac{\partial F}{\partial u_1'}\right)\left(\frac{\partial G}{\partial u_1}\right)^{-1} = \left(\frac{\partial F}{\partial u_k} - \frac{d}{d\,x}\frac{\partial F}{\partial u_k'}\right)\left(\frac{\partial G}{\partial u_k}\right)^{-1}, \quad k = 1,\ldots,n$$

for $u_1,\ldots u_n$. This system, supplemented with the constraint $G(x,u) = 0$, allows for determination of n minimizers $u_1,\ldots u_n$.

The general case of several constraints is considered similarly. Euler equation forms a linear system for vector-function μ; it can be excluded from the system.

A pointwise constraint in the form of inequality Consider the constraint in the form of pointwise inequality:

$$G(x,u) \leq 0, \quad u = u(x), \quad \forall x \in [a,b] \tag{48}$$

This type of constraints is also taken into account by Lagrange function $\mu(x) \geq 0$. The consideration is similar to the arguments used for isoperimetric and finite-dimensional problems with inequality constraints. The stationarity conditions are (compare with (47)

$$\left(\frac{d}{dx}\frac{\partial}{\partial u'} + \frac{\partial}{\partial u}\right)(F + \mu G) = 0, \tag{49}$$

$$G(x,u) \leq 0, \quad \mu(x) \geq 0, \quad \forall x \tag{50}$$

$$\mu G(x,u) = 0: \qquad \text{Either } \mu = 0,\ G < 0 \text{ or } \mu > 0,\ G = 0. \tag{51}$$

A pointwise constraint in the form of differential equation This constraint is considered by a similar technique. An example will be given in the next note.

3.2 Geodesics on a sphere

Geodesics on a surface is the shortest path between two points A and B on it. Let us find a geodesics on a sphere, or the shortest path bewteen A and B.

Assume that the path that joints points A and B is represented in parametric form

$$\begin{gathered}(x(s), y(s), z(s)), \quad s \in [s_0, s_1], \\ (x(s_0), y(s_0), z(s_0)) = A,\ (x(s_1), y(s_1), z(s_1)) = B\end{gathered} \tag{52}$$

The path lies on a sphere:

$$G\left(x(s), y(s), z(s)\right) = 0 \quad \text{where } G(x, y, x) = x^2 + y^2 + z^2 - R^2. \tag{53}$$

.

The distance along the trajectory is

$$\int_{s_0}^{s_1} P(x', y', z')ds, \; P = \sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2} \text{ subject to } (53, 52). \tag{54}$$

where x', y', z' are the derivatives: $x' = \frac{dx}{ds}$, $y' = \frac{dy}{ds}$, $z' = \frac{dz}{ds}$. Thus, finding a geodesics is a pointwise constrained variational problem. Introduce the Lagrange function $\mu(s)$ and write the extended functional as

$$\min_{x(s), y(s), z(s)} \int_{s_0}^{s_1} \left(P(x', y', z') + \mu(s) G(x, y, z)\right) ds; \tag{55}$$

the boundary values (52) are fixed.

Transformations and the solution The Lagrangian is:

$$P(x', y', z') + \mu(s) G(x, y, z) \tag{56}$$

The Euler-Lagrange equation with respect of variation of the function $x(s)$ is

$$\frac{d}{ds}\frac{x'}{P} - 2\mu x = 0$$

and similar for y and x We find μ from these equations

$$\frac{1}{2x}\frac{d}{ds}\left(\frac{x'}{P}\right) = \frac{1}{2y}\frac{d}{ds}\left(\frac{y'}{P}\right) = \frac{1}{2z}\frac{d}{ds}\left(\frac{z'}{P}\right) = \mu$$

The first two equalities are transformed to

$$y\frac{x''P - x'P'}{P^2} = x\frac{y''P - y'P'}{P^2}$$

and

$$x\frac{y''P - y'P'}{P^2} = y\frac{z''P - z'P'}{P^2}$$

Rearranging terms, we find

$$\frac{P'}{P} = \frac{x''y - y''x}{x'y - y'x} = \frac{z''y - y''z}{z'y - y'z} \tag{57}$$

Note that the expressions in the numerators of the second and third terms are full derivatives of the denominators of these terms:

$$\frac{x''y - y''x}{x'y - y'x} = \frac{(x'y - y'x)'}{x'y - y'x} = [\ln(x'y - y'x)]'$$

and similarly

$$\frac{z''y - y''z}{z'y - y'z} = [\ln(z'y - y'z)]'$$

Remark 3.1 (Reminder) Recall definition of logarithmic derivative:

$$\frac{f'(s)}{f(s)} = [\ln f(s)]'$$

Recall also that integration of the relation $[\ln a(s)]' = [\ln b(s)]'$ where $a(s)$ and $b(s)$ are some positive differentiable functions, implies $\ln a(s) = \ln b(s) + \ln C$ where $C > 0$ is a constant; exponentiation then gives $a(s) = Cb(s)$

The last two terms in (57) are logarithmic derivatives. We write the second equality in (57) as

$$[\ln(x'y - y'x)]' = [\ln(z'y - y'z)]'.$$

Integration gives

$$(x'y - y'x) = C_1(z'y - y'z) \tag{58}$$

Rearranging terms in (58) once more, we write

$$\frac{y'}{y} = \frac{[x - C_1 z]'}{x - C_1 z} \text{ or } [\ln(y)]' = [\ln(x - C_1 z)]'$$

Integrating again, we find $\ln(y) = \ln(x - C_1 z) + \ln C_2$; exponentiation shows that x, y, z are linearly related

$$y = C_2 x - C_1 C_2 z.$$

The found optimal trajectory (geodesics) lies on the surface of the sphere and on a plane that passes through the origin (the center of the sphere). The constants of integration C_1 and C_2 are used to fix the position of that plane. A geodesics goes along the meridian that passes through the points A and B.

Remark 3.2 There are two paths that connect the points A and B along the meridian; one of them (the optimal one) is shorter than πR, the other one is longer.

3.3 Shortest path around an obstacle

Let us find the shortest path between two points on a plane assuming that there is an obstacle between these points and the path must go around the obstacle. The coordinates of the points are set to be $A = (0, 0)$ and $B = (d, 0)$. The obstacle O is described as

$$\begin{aligned} O = \{(x, y) \ : \ \phi_1(x) < y < \phi_2, \ x \in [a, b] \\ \phi_1(a) &= \phi_1(b) = 0; \quad \phi_1(x) \leq 0, \ \forall x \in (a, b) \\ \phi_2(a) &= \phi_2(b) = 0; \quad \phi_2(x) \geq 0, \ \forall x \in (a, b)\}, \end{aligned}$$

$\phi_1(x)$ and $\phi_2(x)$ are differentiable functions. We are searching for the path $y(x)$ that minimizes the distance and does not belong to O:

$$D(y) = \int_0^d \sqrt{1 + [y'(x)]^2} dx \quad (x, y) \notin O \tag{59}$$

There are two paths that deliver a local minimum to (59), one $y_1(x) \le 0$ that goes below O

$$y_1(x) - \phi_1(x) \le 0, \forall x \in (a,b) \tag{60}$$

and the other $y_2(x) \ge 0$ goes above O

$$y_2(x) - \phi_2(x) \ge 0, \forall x \in (a,b)$$

These paths should be found independently of each other and their lengths should be compared. We show how to find the best path $y_1(x)$, the other one is found similarly. The variational problem (60) is of the type(48), it is constrained by pointwise inequalities.

Analysis The augmented Lagrangian is

$$L = \sqrt{1 + [y_1']^2} + \lambda(y_1 - \phi_1), \quad \lambda \ge 0.$$

where λ is the Lagrange function by the inequality constraint (60). The stationarity conditions are

$$\frac{d}{dx}\Theta(y_1') - \lambda = 0, \quad \lambda \ge 0 \tag{61}$$

$$\lambda(y_1 - \phi_1) = 0. \tag{62}$$

where

$$\Theta(y') = \frac{y_1'}{\sqrt{1 + [y_1']^2}}$$

We find from (61) and (62) that

$$\begin{array}{ll} \Theta \text{ is constant} & \text{if } \lambda = 0,\ y_1(x) < \phi_1(x) \\ \Theta \text{ increases} & \text{if } \lambda > 0,\ y_1(x) = \phi_1(x) \end{array}$$

Relations (61) show that Θ monotonically increases with x. Notice that $\Theta(y')$ is a monotonically increasing function of y', therefore $y'(x)$ also monotonically increases with x. This means that $y(x)$ is *convex* in (a,b). The second derivatiive $[y'(x)]'$ is nonegative, $y(x)'' \ge 0$.

Equality (62) states that if $y_1(x) < \phi_1(x)$ then $\lambda = 0$ and $y_1' =$ constant: The optimal trajectory is a straight line if $y'(x)$ does not coincide with $\phi_1(x)$. The optimal trajectory satisfies the relations

$$y_1'(x) = \begin{cases} \text{constant} & \text{if } y_1 < \phi_1 \\ \text{increases} & \text{if } y_1 = \phi_1 \end{cases} \tag{63}$$

In addition, (63) also shows that $y_1'(x_s) = \phi_1'(x_s)$ in the points x_s where $y_1(x)$ touches $\phi_1'(x)$ and that $\phi_1'(x_s)$ is convex in the segments where $y_1 = \phi_1$

We conclude that the optimal path $y_1(x)$ is the convex envelope of $\phi_1(x)$ on $[a,b]$. Convex envelope is the maximal of all convex functions that do not exceed ϕ_1. We denote it as

$$y_1 = \mathcal{C}\phi_1, \quad x \in [a,b].$$

A similar argument shows that the negative of the optimal upper path y_2 is the convex envelope of the negative of ϕ_2, or

$$y_2 = -\mathcal{C}(-\phi_2), \quad x \in [a, b].$$

Finally, we need directly compare the lengths $D(y_1)$ and $D(y_2)$, see (59), to determine which of them deliver the global minimum of D.