Infinitely often oscillating solutions

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1 Relaxation of Nonconvex Problems with scalar minimizer

If Lagrangian $F(x, u, u')$ that is nonconvex with respect to the third argumentu' at some part of the stationary trajectory $u(x)$, the Weierstrass test fails, and the stationary trajectory cannot be optimal. Here we investigate variational problems with such Lagrangians.

Consider the Lagrangian as a function of three real arguments $F(x, u, z)$ that is

(a) a nonconvex function of its third argument;

(b) is bounded from below (say, by zero),

$$F(x, u, z) \geq 0 \quad \forall x, u, z; \tag{1}$$

(c) grows superlinearly:

$$\lim_{|z| \to \infty} \frac{F(x, \boldsymbol{y}, \boldsymbol{z})}{|z|} = \infty.$$

Then the infimum I_0

$$I_{\text{inf}} = \inf_u J(u), \quad J(u) = \int_a^b F(x, u, u')dx$$

is nonnegative, $I_{\text{inf}} \geq 0$ and the minimizer has a bounded derivative.

We can construct a minimizing sequence $\{u^s\}$ such that $I(u^s) \to I_{\text{inf}}$. The minimizing sequence $\{u^s\}$ consists of continuous functions with L_1-bounded derivatives; see [**?**].

The differentiable minimizer (if it exists) is a solution to the Euler equation. Also, it satisfies an independent inequality, the Weierstrass test. The inequality states that the Lagrangian at $F(.,.,z)$ the optimal trajectory ($u(x)$ is a convex function of its last argument $z = u'$. Because $F(.,.,z)$ is not convex, this minimizing sequence cannot tend to a differentiable function in the limit. The derivative u' of a minimizer cannot belong to intervals of non-convexity (we call them "forbidden intervals"); it is not immediately clear how this can be achieved because Euler equation does not leave freedom to choose the derivation of the solution.

Recall that the Weierstrass test computes an effect of adding a local perturbation to a smooth stationary minimizer. The perturbation is an infinitesimal zig-zag of the trajectory. If such perturbation decreases the cost of the problem, the minimizer fails the test. We demonstrate that a minimizing sequence tends to a "generalized curve" that consists of infinitesimal zigzags. The limiting function $u(x)$ has a dense set of points of discontinuity of the derivative. Here we give a brief description of it, mainly by working on several examples.

Figure 1: Graph of nonconvex $G(v)$

1.1 A non-convex problem

Consider a simple variational problem that yields to the generalized solution

$$I_{\rm inf} = \inf_u J(u), \quad J(u) = \inf_u \int_0^1 [u^2 + G(u')]dx, \quad u(0) = u(1) = 0 \tag{2}$$

where

$$G(v) = \begin{cases} (v-1)^2, & \text{if } v \geq \frac{1}{2} \\ \frac{1}{2} - v^2 & \text{if } -\frac{1}{2} \leq v \leq \frac{1}{2} \\ (v+1)^2 & \text{if } v \leq -\frac{1}{2} \end{cases} . \tag{3}$$

The graph of the function $G(v)$ is presented in Figure 1. The Lagrangian $F = u^2 + G(u')$ penalizes the trajectory $u(x)$ for having the magnitude $|u'(x)|$ of the derivative different from one and also penalizes the deflection of the trajectory $u(x)$ from zero. These contradictory requirements cannot be resolved in the class of classical trajectories. Indeed, a differentiable minimizer satisfies the Euler equation

$$\begin{array}{ll} u'' - u = 0 & \text{if} \quad |u'| \geq \frac{1}{2} \\ u'' + u = 0 & \text{if} \quad |u'| \leq \frac{1}{2}. \end{array} \tag{4}$$

The Lagrangian $F(u, u')$ is nonconvex in the interval $u' \in (-1,\ 1)$ (see **??**). The Weierstrass test that requires convexity of $G(v)$ supplements the Euler equation (25) with the inequality

$$u' \notin (-1,\ 1) \quad \text{at the optimal trajectory.} \tag{5}$$

Euler equation does not allow to choose the trajectory that satisfes (5) avoiding the forbidden interval.

Remark 1.1 Notice that the second regime in 2.2.4 is never optimal because it is realized inside of the forbidden interval $u' \in (-1,\ 1)$. Moreover, the form of Lagrangian in the whole interval of non-convexity can be arbitrarily altered as long as it remain non-convex; such alternation does not affect the minimizer.

Minimizing sequence We constuct a minimizing sequence for problem (2) without a reference to Euler equation. The infimum of (2) is nonnegative, $\inf_u I(u) \geq 0$. Therefore, any sequence u^s such that

$$\lim_{s\to\infty} I(u^s) = 0 \tag{6}$$

is a minimizing sequence.

(i) Consider a set $\tilde{U}$ of functions $\tilde{u}^s(x)$ that belong to the boundary of the *forbidden interval* (5) of nonconvexity of $G(., u')$; the derivative $\tilde{u}'(x)$ of these function is equal to ± 1:

$$\tilde{U} = \{\tilde{u}'(x) : \ \tilde{u}'(x) = \pm 1 \quad \forall x$$

Figure 2: f02.3

The functions $\tilde{u}^s(x)$ make The second term in the Lagrangian (3) vanish,

$$G(\tilde{u}') = \min\{(\tilde{u}'-1)^2,\ (\tilde{u}'+1)^2\} = 0, \quad \forall \tilde{u}' \in \tilde{U}$$

and the problem becomes

$$I(\tilde{u}^s, (\tilde{u}^s)') = \int_0^1 (\tilde{u}^s)^2 dx. \tag{7}$$

(ii) Next, construct the minimizing sequence: The first term of it is a triangle

$$\tilde{u}^{(1)}(x) = \begin{cases} x, & x \in [0, \frac{1}{2}] \\ 1-x, & x \in [\frac{1}{2}, 1] \end{cases}.$$

We compute the cost of the problem J and the range of $u(x)$

$$J(\tilde{u}^{(1)}) = \frac{1}{2}, \quad \tilde{u}^{(1)}(x) \in \left[0, \frac{1}{2}\right]$$

The second term $\tilde{u}^{(2)}(x)$ consists of two sequential triangles in twice smaller scale

$$\tilde{u}^{(2)}(x) = \begin{cases} \tilde{u}^{(1)}(\frac{x}{2}), & x \in [0, \frac{1}{2}] \\ \tilde{u}^{(1)}(\frac{x}{2} - \frac{1}{2}), & x \in [\frac{1}{2}, 1] \end{cases}$$

The area of each of the two triangles is four times smaller that the area of the triangle in the first term, therefore

$$J(\tilde{u}^{(2)}) = \frac{1}{2} J(\tilde{u}^{(1)}) = \frac{1}{4} \quad \tilde{u}^{(2)}(x) \in \left[0, \frac{1}{4}\right]$$

Continuing this procedure, we construct $\tilde{u}^{(n)}(x)$ as the sequence of n triangles, We have

$$J(\tilde{u}^{(n)}) = \frac{1}{2n}, \quad \tilde{u}^{(n)}(x) \in \left[0, \frac{1}{2n}\right] \quad \forall x \tag{8}$$

The term $\tilde{u}^s$ oscillates near zero if the derivative $(\tilde{u}^s)'$ changes its sign on intervals of equal length. Cost $J(\tilde{u}^s)$ depends on the density of switching points and tends to zero when the number of these points increases (see (8). The minimizing sequence consists of the saw-tooth functions $\tilde{u}^s$; the heights of the teeth tend to zero, and their number tends to infinity as $s \to \infty$.

Oscillating minimizing sequencef2.30.4

Note that minimizing sequence $\{\tilde{u}^s\}$ does not converge to any classical function but rather to a distribution. This minimizer $\tilde{u}^s(x)$ satisfies the contradictory requirements, namely, the derivative must keep the absolute value equal to one, but the function itself must be arbitrarily close to zero:

$$|(\tilde{u}^s)'| = 1 \quad \forall x \in [0,1], \qquad \max_{x \in [0,1]} |\tilde{u}^s| \to 0 \quad \text{as } s \to \infty. \tag{9}$$

The limit has zero norm in $C_0[0,\ 1]$ but a finite norm in $C_1[0,\ 1]$.

$$\lim_{s\to\infty}\int_0^1 (\tilde{u}^s)^2 dx = 0, \quad \lim_{s\to\infty}\int_0^1 (\tilde{u'}^s)^2 dx = 1, \quad \lim_{s\to\infty}\int_0^1 \tilde{u'}^s dx = 0$$

Notice that $u'(x)$ in this solution takes one of to the values $u'(x) = \pm 1$; these values are the supporting points of the convex envelope $\mathcal{C}G(u')$ of $G(u')$, or the point where the envelope touches $G(u')$:

$$\mathcal{C}G(u') = G(u'), \quad 0 = \frac{d}{du'}\mathcal{C}G(u') = \frac{d}{du'}G(u')$$

1.2 Relaxed problem

The same type of oscillating solution occurs in a more general problem with non-convex Lagrangian. Consider Lagrangian $F(x, u, u')$ where $u(x)$ is a scalar function. Assume that it is bounded from below, is of superlinear growth with respect to u', and that it is non-convex function of u' in a finite interval $u' = [\rho_1(u,t), \rho_2(u,t)]$. The Weierstrass test is not satisfied for u' is the forbidden interval, therefore

$$u' \notin [\rho_1, \rho_2]$$

When u' reaches an endpoint of this interval, it must instantly jump to the other end of it. At the jump point, the Lagrangian satisfies Weierstrass-Erdmann condition

$$\left.\frac{\partial F}{\partial u'}\right|_{u'=\rho_1} = \left.\frac{\partial L}{\partial u'}\right|_{u'=\rho_2}$$

After u' jumps from ρ_1 to ρ_2, it may jump back, and again. When the jumps occur infinitely fast, the minimizer becomes a generalized curve.

Relaxation We describe a generalized solution passing to the homogenized variables or the averages over a small interval. Consider a small interval $x = [x_0, x_0 + \epsilon]$ and a function $u_\epsilon(x)$ with ϵ-periodic derivative. Assume that derivative $u'_\epsilon(x)$ takes two values

$$u'_\epsilon(x) = \begin{cases} \rho_1, \ x \in [x_0, x_0 + m\epsilon) \\ \rho_2, \ x \in [x_0 + m\epsilon, x_0 + \epsilon) \end{cases}$$

where $m \in [0, 1]$ is a length of the fraction of the interval where $u'_\epsilon(x) = \rho_1$.

The average (homogenized) value $v_h(x_0)$ of the derivative at the interval $[x_0, x_0 + \epsilon]$ is

$$v_h(x_0) = \frac{1}{\epsilon}\int_{x_0}^{x_0+\epsilon} u'_\epsilon(x)dx = m\rho_1 + (1-m)\rho_2, \quad v_\epsilon(x) \in [\rho_1, \rho_2];$$

it depends of $\rho_1,\ \rho_2$ and the fraction $m \in [0, 1]$. The homogenized derivative $u'_h(x)$ is the convex combination of derivatives ρ_1 and ρ_2 at the end point of the interval of non-convexity.

The minimizer $u_\epsilon(x)$ is a zig-zag curve. Its value at $(x+\epsilon) =$ depends only on the average derivative v_h of the discontinuous derivative $u_h'(x)$:

$$u_h(x+\epsilon) = u_h(x) + \epsilon v_h(x_0).$$

or, equivalently,

$$\frac{u_h(x+\epsilon) - u_h(x)}{\epsilon} = v_h(x_0).$$

If $u_h'(x) \in [\rho_1, \rho_2]$, the average $v(x)$ approximates $u_0'(x)$ by the proper choice of m:

The L_2-norm of the difference between the minimizer and its approximation

$$\int_a^b \left(u_\epsilon(x) - u_h(x)\right)^2 dx = O(\epsilon)$$

is infinitesimally small when $\epsilon \to 0$.

The value of the cost J in an interval $[x = x_0, x_0 + \epsilon]$ is

$$J(x_0, \epsilon) = \int_{x_0}^{x_0+\epsilon} \hat{F} dx = \epsilon \left[\int_{x_0}^{x_0+m\epsilon} F(x, u, \rho_1) + \int_{x_0+m\epsilon}^{x_0+\epsilon} F(x, u(x), \rho_2) + O(\epsilon) \right]$$

When $\epsilon \to 0$, the cost tends to the convex combination of $F(x, u, \rho_1)$ and $F(x, u(x), \rho_2)$:

$$J(x_0, \epsilon) = \epsilon \left[m F(x, u, \rho_1) + (1-m) F(x, u(x), \rho_2) + O(\epsilon) \right]$$

Optimizing $J(x_0, \epsilon)$ with respect of ρ_1, ρ_2, and m, we find the best value of the cost at the zigzag minimizers:

$$\begin{aligned} I(x_0, \epsilon) &\geq \min_{m \in \mathcal{M}} \min_{\rho_1, \rho_2} \epsilon \left[m F(x, u, \rho_1) + (1-m) F(x, u(x), \rho_2) \right] \\ \mathcal{M} &= \{ m \in [0,1], m\rho_1 + (1-m)\rho_2 = u' \} \end{aligned}$$

The right-hand side of the above inequality is the convex envelope $\mathcal{C}_v F$ of $F(x, u, v)$ with respect to its third argument $v = u'$.

This way, we define the *relaxed* problem

$$I(x) \geq I_R(u); \quad I_R(u) = \min_u \int_a^b \mathcal{C}_{u_h'} F(x, u_h, u_h') dx$$

Properties of the Relaxed Problem

- Recall that the derivative of the minimizer never takes values in the region $\mathcal{Z}_\mathrm{f}$ of nonconvexity of F. Therefore, a solution to a nonconvex problem stays the same if its Lagrangian $F(x, u, z)$ is replaced by any Lagrangian $\mathcal{N}F(x, u, z)$ that satisfies the restrictions

$$\begin{aligned} \mathcal{N}F(x, u, z) &= F(x, u, z) \quad \forall\ z \notin \mathcal{Z}_\mathrm{f}, \\ \mathcal{N}F(x, u, z) &> \mathcal{C}F(x, u, z)\ \forall\ z \in \mathcal{Z}_\mathrm{f}. \end{aligned} \tag{10}$$

Indeed, the two Lagrangians $F(x,u,z)$ and $\mathcal{N}F(x,u,z)$ coincide in the region of convexity of F. Therefore, the solutions to the variational problem also coincide in this region. Neither Lagrangian satisfies the Weierstrass test in the forbidden region of nonconvexity. Therefore, no minimizer can distinguish between these two problems: It never takes values in Z_{f}. The behavior of the Lagrangian in the forbidden region is simply of no importance. In this interval, the Lagrangian cannot be computed from the minimizer.

- The infimum of the functional for the initial problem coincides with the minimum of the functional in the relaxed problem. The Lagrangian in the relaxed problem is convex. The Weierstrass condition is satisfied, and the minimal solution (if it exists) is stable against fine-scale perturbations. To be sure that the solution of the relaxed problem exists, one should also examine other sources of possible nonexistence (see, for example, [**?**]).

- The number of minimizers in the relaxed problem is increased in the relaxed problem. Instead of one minimizer $u(x)$ in the original problem, the minimizers consist of three "slow-varied" minimizers $\rho_1(x), \rho_2(x)$ and $m(x)$ bounded by the equality (**??**) and the inequalities $m(x) \in [0,1]$. They define the derivative

$$u' = m(x)\rho_1(x) + (1 - m(x))\rho_2(x)$$

of the relaxed minimizer. The relaxed problem is controlled by these independent parameters that determine the alternating minimizing sequence.

- In the forbidden region, Euler equation degenerates. If the convex envelope does not coincide with G, it has the form

$$\mathcal{C}F = au' + b(x,u)$$

This representation implies that the first term in the left-hand side (**??**) of the Euler equation (**??**) vanishes identically: $\frac{d}{dx}\frac{\partial}{\partial u'}\mathcal{C}F \equiv 0$. The Euler equation degenerates into an algebraic equation $\frac{\partial}{\partial u} b(x,u) = 0$. Optimal fraction m can be found from the equation

$$\frac{d}{dx}\frac{\partial}{\partial u} b(x,u) = \frac{\partial^2 b(x,u)}{\partial x \partial u} + \frac{\partial^2 b(x,u)}{\partial u^2} u' = 0$$

1.3 Examples: Solutions to Nonconvex Problems

A Two-Well Lagrangian We turn to a more advanced example of the relaxation of an ill-posed nonconvex variational problem. This example highlights more properties of relaxation and introduces piecewise quadratic Lagrangians.

Example 1.1 Consider the minimization problem

$$\min_{u(x)} \int_0^z F(x,u,u'), \quad u(0) = 0, \; u'(z) = 0 \tag{11}$$

Figure 3: f02.4

with a Lagrangian

$$F = (u - \gamma x^2)^2 + G(u'), \tag{12}$$

where

$$G(v) = \min\{a\, v^2,\ b\, v^2 + 1\}, \quad 0 < a < b,\ \gamma > 0.$$

The first term $(u - \gamma x^2)^2$ of the Lagrangian forces the minimizer u and its derivative u' to increase with x, until u' at some point reaches the interval of nonconvexity of $G(u')$, to pass this interval, and increase further. The term G is a nonconvex function of u'. The derivative u' stays outside of the forbidden interval of nonconvexity of the function G. The convex envelope $\mathcal{C}G(v)$ of $G(v)$ is (see Example 21)

$$\mathcal{C}G(v) = \begin{cases} a\, v^2 & \text{if } \ |v| \le v_1, \\ 2v\sqrt{\frac{ab}{a-b}} - \frac{b}{a-b} & \text{if } \ v_1 \le |v| \le v_2, \\ b(v)^2 + 1 & \text{if } \ |v| \ge v_2. \end{cases}$$

where

$$v_1 = \sqrt{\frac{b}{a(a-b)}}, \quad v_2 = \sqrt{\frac{a}{b(a-b)}},$$

Convexification of the Lagrangian and the minimizer f2.4 0.4

The relaxed problem has the form

$$\min_u \int \mathcal{C}F(x, u, u')dx, \tag{13}$$

where

$$\mathcal{C}F_L(x, u, u') = \begin{cases} (u - \gamma x^2)^2 + a(u')^2 & \text{if } \ |u'| \le v_1, \\ (u - \gamma x^2)^2 + 2u'\sqrt{\frac{ab}{a-b}} - \frac{b}{a-b} & \text{if } \ v_1 \le |u'| \le v_2, \\ (u - \gamma x^2)^2 + b(u')^2 + 1 & \text{if } \ |u'| \ge v_2. \end{cases}$$

Recall that the variables u, u' in the relaxed problem are the averages of the original variables; they coincide with those variables everywhere when $\mathcal{C}F = F$. The Euler equation of the relaxed problem is

$$\begin{cases} au'' - (u - \gamma x^2) = 0 & \text{if } \ |u'| \le v_1, \\ (u - \gamma x^2) = 0 & \text{if } \ v_1 \le |u'| \le v_2, \\ bu'' - (u - \gamma x^2) = 0 & \text{if } \ |u'| \ge v_2. \end{cases} \tag{14}$$

where v_1 and v_2 are defined in (**??**). The boundary conditions are shown in (11).

Notice that the Euler equation degenerates into an algebraic equation in the interval where convex envelope of F does not coincide with F.

Integrating the Euler equations, we sequentially meet all three regimes when both the minimizer and its derivative monotonically increase with x (see **??**). If the length z of the interval of integration is sufficiently large, one sees all three regimes.

Minimizing sequence Let us describe minimizing sequences that form the solution to the relaxed problem. Recall that the actual optimal solution is a generalized curve in the region of nonconvexity; this curve consists of infinitely often alternating parts with the derivatives v_1 and v_2 and the relative fractions $m(x)$ and $1-m(x)$, respectively:

$$u'(x) = m(x)v_1 + (1-m(x))v_2, \quad u' \in [v_1, v_2], \tag{15}$$

The Euler equation degenerates in the second region into an algebraic one $\langle u \rangle = \gamma x^2$ because of the linear dependence of the Lagrangian on $\langle u \rangle'$ in this region. The first term of the Euler equation,

$$\frac{d}{dx}\frac{\partial F}{\partial \langle u \rangle'} \equiv 0 \quad \text{if} \quad v_1 \le |\langle u \rangle'| \le v_2, \tag{16}$$

vanishes.

Obtaining optimal fraction m The variable $m(x)$ of the generalized solution is found by differentiation of the optimal solution:

$$(u(x) - \gamma x^2)' = 0 \quad \text{or} \quad u'(x) = 2\gamma x. \tag{17}$$

Using definition (15) of the average derivative, we find

$$u'(x) = m(x)v_1 + [1-m(x)]v_2 = 2\gamma x.$$

(recall that the boundaries v_1 and v_2 of the forbidden interval are constant in this problem). Solving the equality for m, we obtain

$$m = \begin{cases} 0 & \text{if} \quad |u'| \le v_1, \\ \frac{2\alpha}{v_1 - v_2}x - \frac{v_2}{v_1-v_2} & \text{if} \quad v_1 \le |u'| \le v_2, \\ 1 & \text{if} \quad |u'| \ge v_2. \end{cases} \tag{18}$$

Variable $m(x)$ linearly increases within the second region (see **??**). Note that the pointwise derivative u' of the minimizing generalized curve belong to one of the boundaries v_1 or v_2 at each point x of the forbidden interval of nonconvexity of F; the average derivative $u'(x)$varies only due to varying of the fraction $m(x)$ (see **??**).

2 Relaxation of problems with vector minimizer

2.1 Relaxation procedure

The procedure is essentially the same. A bounded from below of Lagrangian $F(x,u,v)$ of superlinear with respect of z growth with a the non-convex with respect of v region is replaced with its convex envelope $F_v(x,u,v)$. Every point of the convex envelope of a function of n-dimensional vector is a convex combination by $n+1$ supporting points ρ_1, ρ_{n+1},

$$v = \sum_{i=1}^{n+1} m_i \rho_i, \quad \sum_{i=1}^{n+1} m_i = 1, \quad m_i \ge 0$$

The minimizing sequence is a fast oscillating vector function $v(x)$ takes not more that $n+1$ values in each infinitesimal interval. The relaxed Lagrangian is

$$RF(x,u,u') = \min_{m_1\ldots m_{n+1}} \left(\min_{\rho_1\cdots\rho_{n+1}} \sum_{i=1}^{n+1} m_i F(x,u,u') \right) \tag{19}$$

$$u' = \sum_{i=1}^{n+1} m_i \rho_i, \quad \sum_{i=1}^{n+1} m_i = 1, \quad m_i \geq 0 \tag{20}$$

Calculating the minima, we express the relaxed Lagrangian through convex envelope with respect to $v = u'$

$$RF(x,u,u') = \mathcal{C}F(x,u_h,u_h') \tag{21}$$

Here u_h is the homogenized (averaged over a small ϵ-interval) minimizer, u_h' is its homogenized derivative.

Instead of one minimizer $u(x)$ in the original problem with nonconvex Lagrangian, the relaxed problem depends on several minimizers: the supporting points $\rho_i(x)$ of the convex envelope and the barycentric coordinates $m_i(x)$. These continuous functions describe parameters of infinitely often oscillating minimizing sequence with the derivative that sequentially takes values $\rho_1,\ldots\rho_n$ in the infinitely small intervals $[x, x+\epsilon]$.

Remark 2.1 Analyzing the homogenized solution, one cannot determine what value takes u' at a specific point x but only the relative length (measure) of the intervals where a specific value is taken. Such fast oscillating sequences are called solutions in Young measures.

2.2 Examples of nonconvex problems for vector minimizer

Three-well Lagrangian Consider the problem with the Lagrangian

$$\mathcal{C}F(v_1,v_2) + \Phi(x,u_1,u_2)$$

where $\mathcal{C}F$ is the convex envelope of three-well function described in example 21.

In the domains where the Lagrangian is convex and the convex envelope $\mathcal{C}F(v_1,v_2)$ coincides with the wells in $F(v_1,v_2)$, stationarity conditions are represented by a system of two second-order Euler equations:

$$2u_1'' - \frac{\partial \Phi}{\partial u_1} = 0, \quad 2u_2'' - \frac{\partial \Phi}{\partial u_2} = 0,$$

Notice that linear with respect to derivative terms in the second and third wells are null-Lagrangians and they do not affect Euler equations, because

$$\frac{d}{dx}\left(\frac{\partial (u'-1)^2}{\partial u'}\right) = 2\frac{d(u'-1)}{dx} = 2u''$$

In the complex hull Ω_0, where

$$\mathcal{C}F(v_1, v_2) = 0$$

stationarity is described by algebraic equations for u_1 and u_2:

$$\frac{\partial \Phi}{\partial u_1} = 0, \quad \frac{\partial \Phi}{\partial u_2} = 0, \tag{22}$$

These minimizers are zigzag functions which derivatives taken pointwize values $\rho_1 = (0,0), \rho_2 = (1,0)$, and $\rho_3 = (0,1)$. The weights (measures) m_i are found by differentiation of the conditions (22) and (20):

$$\frac{d}{dx}\frac{\partial \Phi}{\partial u_1} = \frac{\partial^2 \Phi}{\partial x \partial u_2} + \frac{\partial^2 \Phi}{\partial u_1^2} u_1' + \frac{\partial^2 \Phi}{\partial u_1 \partial u_2} u_2' \tag{23}$$

$$\frac{d}{dx}\frac{\partial \Phi}{\partial u_2} = \frac{\partial^2 \Phi}{\partial x \partial u_2} + \frac{\partial^2 \Phi}{\partial u_1 \partial u_2} u_1' + \frac{\partial^2 \Phi}{\partial u_2^2} u_2' \tag{24}$$

This two equations are linear relations for m_1, m_2, m_3 because

$$u' = \sum_{i=1}^{3} m_i \rho_i;$$

together with the third equation $m_1 + m_2 + m_3 = 1$, they allow for finding barycentric coordinates m_i.

In the remaining domains, stationarity conditions include one second-order differential equation and one algebraic equation. For example, in the domain Ω_1, the relaxed Lagrangian is

$$F_2 = (u_2')^2 + \Phi, (x, u_1, u_2), \quad u_1' \in [0,1]$$

the Euler equations are

$$\frac{\partial \Phi}{\partial u_1} = 0 \quad 2u_2'' = \frac{\partial \Phi}{\partial u_2} = 0$$

Barycentric coordinates $m_1, m_2,\ m_1 + m_2 = 1$, are found by differentiation of the first stationarity equation as in (24)

$$\frac{d}{dx}\frac{\partial \Phi}{\partial u_1} = \frac{\partial^2 \Phi}{\partial x \partial u_1} + \frac{\partial^2 \Phi}{\partial u_1^2} u_1' = 0$$

Express u_1' as function of m_1: $u_1' = m_1 \rho_1 + m_2 \rho_2$ where $Gr_1 = 0, Gr_2 = 1$ and $m_2 = 1 - m_1$: $u_1' = m_2$ and the stationarity conditions, we find

$$m_2 = -\frac{\partial^2 \Phi}{\partial x \partial u_1} \left(\frac{\partial^2 \Phi}{\partial u_1^2} \right)^{-1}$$

The other two cases are treated similarly.

We can also check that determinant of the Hessian is zero everywhere, where $\mathcal{C}F < F$.

2.3 Conclusion and Problems

We have observed the following:

- A variational problem has the fine-scale oscillatory minimizer if its Lagrangian $F(x, u, u')$ is a nonconvex function of its third argument.
- Homogenization leads to the relaxed form of the problem that has a classical solution and preserves the cost of the original problem.
- The relaxed problem is obtained by replacing the Lagrangian of the initial problem by its convex envelope. It can be computed as the second conjugate to F.
- The dependence of the Lagrangian on its third argument in the region of nonconvexity does not affect the relaxed problem.

To relax a variational problem, we use two ideas. First, we replaced the function with its convex envelope and got a stable extension of the problem. Second, we proved that the value of the integral of the convex envelope $\mathcal{C}F(\boldsymbol{v})$ of a given function is equal to the value of the integral of this function $F(\boldsymbol{v})$ if its argument $\boldsymbol{v}$ is a zigzag curve. We use the Carathéodory theorem, which tells that the number of subregions .whe constancy of the argument is less than or equal to $n+1$, where n is the dimension of the minimizer.

Regularization and relaxation The considered nonconvex problem is another example of an ill-posed variational problem. For these problems, the classical variational technique based on the Euler equation fails to work. Here, The limiting curve is not a discontinuous curve as in the previous example, but a limit of infinitely fast oscillating functions, similar to $\lim_{\omega\to\infty}\sin(\omega x)$.

We may apply regularization to discourage the solution to oscillate. Doing this, we pass to the problem

$$\min_u \int_0^1 (\epsilon^2 (u'')^2 + G(u, u'))dx$$

that corresponds to Euler equation:

$$\begin{array}{ll} \epsilon^2 u^{IV} - u'' + u = 0 & \text{if} \quad |u'| \geq \frac{1}{2} \\ \epsilon^2 u^{IV} + u'' + u = 0 & \text{if} \quad |u'| \leq \frac{1}{2}. \end{array} \tag{25}$$

The Weierstrass condition this time requires the convexity of the dependence of Lagrangian on u''; this condition is satisfies.

The solution of Euler equations is oscillatory, with the period of oscillation of the order of ϵ. It $\epsilon \to 0$, the solution still tends to an infinitely often oscillating distribution. When ϵ is positive but small, the solution has a finite but large number of wiggles. The computation of such solutions is difficult and some times unnecessary: It strongly depends on an artificial parameter ϵ, which is difficult to justify physically. It is n=more natural to replace an ill-posed problem with a *relaxed* one. The idea of relaxation is in a sense opposite to the regularization. Instead of discouraging fast oscillations, we admit them as legitimate minimizers and describe such minimizers in terms of smooth functions: the limits of oscillating variable and the average time that it spends on each boundary.