Hamiltonian

February 28, 2019

Contents

1 Canonical system and Hamiltonian

In this section, we bring Euler equations to the standard form using a modified form of Lagrangian.

1.1 Canonical form of Euler equations

The Euler equations for a vector minimizer $u = (u_1, \ldots, u_N)$ is a system of N second order differential equations:

$$\frac{d}{dx}\frac{\partial L}{\partial u_i'} - \frac{\partial L}{\partial u_i} = 0, \quad i = 1, \ldots, N \tag{1}$$

with boundary conditions

$$\Theta_a(u, u')|_{x=a} = 0, \quad \Theta_b(u, u')|_{x=b} = 0 \tag{2}$$

where Θ_a and Θ_b are some N-dimensional vector functions.

The structure of this system can be simplified and unified if instead of N second-order equations the system is rewritten as a system of $2N$ first order differential equations in a standard form

$$z_i = Y_i(z_1, \ldots, z_{2N}), \quad i = 1, \ldots, 2N$$

where $z(x)$ is a $2N$-dimensional vector of unknowns and Y is a vector-valued function of z.

This system can be obtained from (1) if new variables p_i are introduced,

$$p_i(x) = \frac{\partial L(x,u,u')}{\partial u'_i}, \quad i = 1, \dots, N \tag{3}$$

Variables p_i are called *dual variables.* In mechanics, $p = (p_1, \dots, p_N)$ is called the vector of *impulse.*

The Euler equation takes the form

$$p' = \frac{\partial L(x,u,u')}{\partial u} = f(x,u,u'), \tag{4}$$

where f is a vector-valued function of x, u, u'. The system (3), (4) becomes symmetric with respect to p and u if we algebraically solve (3) for u':

$$u' = \phi(x,u,p). \tag{5}$$

Then, substitute this expression into (4) and obtain:

$$p' = f(x,u,\phi(x,u,p)) = \psi(x,u,p) \tag{6}$$

where ψ is a function of the variables u and p but not of their derivatives. Equations (5), (6) form the canonical system of $2N$ equations for $2N$ unknown functions u_i, p_j, $i, j = 1, \dots, N$.

The boundary conditions (2) are rewritten in terms of u and p excluding u' using (5); they take the form

$$\Theta_a(u, \phi(a,u,p),)|_{x=a} = \theta_a(u,p) = 0, \quad \Theta_b(u,u')|_{x=b} = \theta_b(u,p) = 0 \tag{7}$$

where θ_a and θ_b are N-dimensional vector functions.

In summary, new variable p, see (3), transforms Euler equation to the canonical form (5), (6) (known also as Cauchy, normal, or standard form):

$$\begin{aligned} u' &= \phi(x,u,p) \\ p' &= \psi(x,u,p) \end{aligned} \tag{8}$$

$$\theta_a(u,p) = 0, \quad \theta_b(u,p) = 0. \tag{9}$$

The solution to the canonical system is entirely determined by the algebraic vector functions ϕ, ψ in the right-hand side which does not contain derivatives, and by the boundary conditions. Notice that functions u and p are differentiable.

Example 1.1 (Quadratic Lagrangian) Assume that Lagrangian L and boundary conditions are:

$$L = \frac{1}{2}a(x)u'^2 + \frac{1}{2}b(x)u^2, \quad u(x_0) = u_0, \quad \frac{\partial L}{\partial u'}|_{x=x_1} = 0$$

The Euler equation

$$(a\,u')' - bu = 0$$

is transformed as follows. We introduce p as in (3)

$$p = \frac{\partial L(x,u,u')}{\partial u'} = au'$$

and obtain the canonical system and boundary conditions

$$\begin{array}{l} u' = \frac{1}{a(x)}p \\ p' = b(x)u \\ u(x_0) = u_0, \quad p(x_1) = 0 \end{array}$$

Notice that the coefficient $a(x)$ is moved into denominator.

1.2 Hamiltonian

We can rewrite the system (8) in a more symmetric form introducing a special potential function called *Hamiltonian.* The Hamiltonian is defined by the formula $u' \frac{\partial L}{\partial u'} - L$ where u' is excluded by the relation $u' = \phi(x,u,p)$:

$$H(x,u,p) = pu' - L(x,u,u') \quad u' = \phi(x,u,p) \tag{10}$$

or

$$H(x,u,p) = p\phi(x,u,p) - L(x,u,\phi(x,u,p)). \tag{11}$$

Here u is a stationary trajectory – the solution of Euler equation.

Hamiltonian allows writing canonical system (8) in a remarkable symmetric form

$$p' = -\frac{\partial H}{\partial u}, \quad u' = -\frac{\partial H}{\partial p} \tag{12}$$

To demonstrate this, compute the partial derivatives of H (11) : We have

$$\frac{\partial H}{\partial u} = p\frac{\partial \phi}{\partial u} - \frac{\partial L}{\partial u} - \frac{\partial L}{\partial \phi}\frac{\partial \phi}{\partial u}$$

By the definition (3) of p, $p = \frac{\partial L}{\partial u'} = \frac{\partial L}{\partial \phi}$, hence the first and third term in the right-hand side cancel. By virtue of the Euler equation (4), the remaining term $\frac{\partial L}{\partial u}$ is equal to p' and we obtain the first equation in (12)

Next, compute $\frac{\partial H}{\partial p}$. We have

$$\frac{\partial H}{\partial p} = \phi + p\frac{\partial \phi}{\partial p} - \frac{\partial L}{\partial \phi}\frac{\partial \phi}{\partial p}$$

By definition of p, the second and the third term in the right-hand side cancel, and by definition of ϕ ($\phi = u'$) we obtain the second equation in (12)

The right-hand side functions in the canonical system (8) are expressed through the partial derivatives of a single potential function $H(u,p)$.

Lagrangian L, Hamiltonian H in Example (1.1) are as follows

$$L = \frac{1}{2}\left(a(x)u'^2 + b(x)u^2\right) = \frac{1}{2}\left(\frac{1}{a(x)}p^2 + b(x)u^2\right)$$

$$H = p\left(\frac{p}{a}\right) - L = \frac{1}{2}\left(\frac{1}{a(x)}p^2 - b(x)u^2\right)$$

the canonical system is

$$\frac{\partial H}{\partial u} = -b(x)u = -p', \quad \frac{\partial H}{\partial p} = \frac{1}{a(x)}p = u'$$

which coincides with the system in Example (1.1).

1.3 The first integrals through the Hamiltonian

System (12) demonstrates that

$$\text{if } H = \text{ constant}(u_i), \quad \text{then } p_i = \text{ constant} \tag{13}$$

and

$$\text{if } H = \text{ constant}(p_i), \quad \text{then } u_i = \text{ constant} \tag{14}$$

These equations correspond to the first integrals in the Euler equation

$$\frac{d}{dx}\frac{\partial L}{\partial u_i'} - \frac{\partial L}{\partial u_i} = 0, \quad i = 1, \ldots, N$$

Indeed,
if Lagrangian L is independent of u_i, $\frac{\partial L}{\partial u_i} = 0$, $\frac{\partial L}{\partial u_i'} =$ constant;
if Lagrangian is independent of u_i', $\frac{\partial L}{\partial u_i'} = 0$, then $\frac{\partial L}{\partial u_i} = 0$. Since $\frac{\partial L}{\partial u_i} = p_i'$, p_i is constant.

Conservative system: Lagrangian is independent of x If $F = F(u, u')$, than

$$H(u, p) = \text{ constant} \tag{15}$$

Indeed, compute the time derivative of the Hamiltonian using the chain rule

$$\frac{d}{dx}H(x, u, p) = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial u}u' + \frac{\partial H}{\partial p}p' = \frac{\partial H}{\partial x}$$

because of equalities (12), $u' = \frac{\partial H}{\partial p}$ and $p' = -\frac{\partial H}{\partial u}$. If Lagrangian does not explicitly depend on x, $\frac{\partial L}{\partial x} = 0$ the Hamiltonian does not explicitly depend on x as well, $\frac{\partial H}{\partial x} = 0$, and we arrive at (15). In mechanics, (15) corresponds to the conservation of the total energy.

Natural boundary conditions The natural or variational boundary conditions that are imposed at the endpoint b from the requirement of minimization of the functional, are $\frac{\partial L}{\partial u'} = 0$ at $x = b$. By definition of the impulse, it is rewritten as

$$p = 0 \quad \text{at } x = b;$$

Transversality condition The transversality condition (25) at the unknown endpoint $x = b$ of the trajectory $u(x)$ is expressed through Lagrangian $L(x, u, u')$ as

$$L - u' \frac{\partial L}{\partial u'} = 0$$

The expression in the left-hand side is Hamiltonian, therefore the condition takes a simple form:

$$H = 0 \quad \text{at } x = b.$$

If the system is conservative, the Hamiltonian is constant (15); therefore, there is no optimal endpoint for such systems.

Weierstrass-Erdmann condition This condition states that at all points of the optimal trajectory, $\frac{\partial L}{\partial u'}$ is continuous. It translates into a statement that impulse p is continuous everywhere. Notice that by virtue of (12), p is differentiable.

Lagrangian and Hamiltonian Both functions describe the same process, but

- Hamiltonian is an algebraic function of differentiable arguments p and u, and Lagrangian is an expression for u, and it's derivative u', the derivative may be discontinuous.

- Optimality conditions for Hamiltonian are expressed as a system of first-order differential equations in canonical form. Optimality conditions for Lagrangian are expressed as a system of second-order differential equations.

- Invariant properties and boundary conditions are more conveniently expressed through Hamiltonian.

- Lagrangian deals with the minimizer and its derivatives; its minimization is a realization of the minimal principle.

2 Examples

2.1 Lagrangian mechanics

Canonical system for equations of Lagrangian mechanics The equations of Lagrangian mechanics correspond to stationarity of the functional

$$L(t, q, q') = T(q, q') - V(q), \quad T(q, q') = \frac{1}{2}(q')^T R(q) q'$$

that callee the action. Here q is the N dimensional vector of generalized coordinates $q = q_1, \ldots, N$, $T(q, q')$ is the kinetic energy, $R(q)$ is a symmetric positively defined matrix of inertia, and the potential energy $V(q)$ is a convex function of q of the N dimensional vector of generalized coordinates $q = q_1, \ldots, N$.

The vector-valued Euler equation

$$\frac{d}{dt}\frac{\partial T}{\partial q'} = \frac{\partial T}{\partial q} - \frac{\partial V}{\partial q} \tag{16}$$

is of order $2N$.

To bring the system (16) to canonical form, we introduce vector of impulses

$$p = \frac{\partial T}{\partial q'} = R(q) q'$$

The Euler equation becomes:

$$p' = \frac{\partial T}{\partial q} - \frac{\partial V}{\partial q}$$

Kinetic energy T is expressed through p as

$$T = \frac{1}{2}(q')^T R q' = \frac{1}{2} p^T \left(R^{-1}\right) p. \tag{17}$$

The first term in the right-hand side of (16) becomes

$$\frac{\partial T}{\partial q} = \frac{1}{2} p^T \left(\frac{dR^{-1}}{dq}\right) p$$

The canonical system becomes

$$\begin{aligned} &q' = R^{-1} p, \\ &p' = \tfrac{\partial T}{\partial q} - \tfrac{\partial V}{\partial q} = \tfrac{1}{2} p^T \left(\tfrac{dR^{-1}}{dq}\right) p - \tfrac{\partial V}{\partial q} \end{aligned}$$

Hamiltonian for Lagrangian mechanics In Lagrangian mechanics, Lagrangian $L = T - V$ implies that Hamiltonian H is the sum of kinetic and potential energy $H = T + V$,

$$H(q, p) = T(q, q') + V(q)$$

where q' is expressed through p and q and $q' = R(q)^{-1}p$. Indeed, we obtain using (17)

$$p^T q' = p^T R^{-1}(q) p = 2T$$

and

$$H = p^T q' - L = 2T - (T - V) = T + V$$

The Hamiltonian is equal to the whole energy of the system.

2.2 Orbiting mass

Consider a point mass m attached by a spring to a fixed point; call this point the origin. The force F in the spring is the derivative of a potential $V(|r|)$, where $r = (r_1, r_2, r_3)$ is the vector of coordinates of a point, $|r| = \sqrt{r_1^2 + r_2^2 + r_3^2}$ is the distance fro the origin. The force F is computed as

$$F = \frac{dV}{dr} = \frac{dV'}{d|r|}\frac{d|r|}{dr} = \left(\frac{dV}{d|r|}\frac{1}{|r|}\right) r = \phi(|r|) r$$

where

$$\phi(|r|) = \frac{dV}{d|r|}\frac{1}{|r|}$$

The Lagrangian is

$$L = T - V = \frac{1}{2} m r'^T r' - V(|r|)$$

Euler equations are

$$(mr')' + \frac{\partial V}{\partial r} = 0 \quad \text{or } (mr')' + \phi(|r|) r = 0$$

Introducing the impulse vector $p = mr'$ and we write canonical system as

$$r' = \frac{1}{m} p, \quad p' = -\phi(|r|) r \tag{18}$$

Planar motion Analyzing system (18), we conclude that the movement is planar. Indeed, consider vector product $z = r \times p$ and compute its time derivative:

$$\frac{dz}{dt} = \frac{d}{dt}(r \times p) = r' \times p + r \times p' = 0$$

This vector is constant because vector r' is proportional to p and p' is proportional to r, see (18). The constancy of z indicates that vectors $r(t)$ and $p(t)$ remain all the time perpendicular to vector z; they are moving in a plane L, that passes through the point of initial conditions $r(0) = r_0,\ p(0) = p_0$ and the origin.

In the plane L, we introduce polar coordinates ρ, θ; Potential energy depends only on ρ, $V = V(\rho)$, and kinetic energy becomes

$$T = \frac{m}{2}(\rho'^2 + \rho^2 \theta'^2) = \frac{1}{2} p^T R^{-1} p$$

where

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad R = \begin{pmatrix} m & 0 \\ 0 & m\rho^2 \end{pmatrix}$$

The canonical system becomes

$$\dot{\rho} = \frac{1}{m} p_1, \tag{19}$$

$$\dot{\theta} = \frac{1}{m\rho^2} p_2, \tag{20}$$

$$\dot{p_1} = -\frac{2}{m\rho^3} p_1 - \frac{\partial V}{\partial \rho} \tag{21}$$

$$\dot{p_2} = 0 \tag{22}$$

Hamiltonian The Hamiltonian is

$$H = T + V = \frac{1}{2m} \left(\dot{\rho}^2 + \frac{1}{\rho^2} \dot{\theta}^2 \right) + V(\rho) \tag{23}$$

One can check that the equations (19)-(22) can be obtained by differentiation of H with respect of ρ, θ, p_1, and p_2

Invariants The Hamiltonian is independent of θ; therefore p_2 is constant, $p_2 = C_1$, see (22), and Hamiltonian becomes function of ρ and p_1 only:

$$H = \frac{1}{2m} (\rho'^2 + \frac{1}{\rho^2} C_1^2) + V(\rho) \tag{24}$$

The Hamiltonian (24) is independent of time t therefore is constant,

$$H = \frac{1}{2m} \left(\rho'^2 + \frac{1}{\rho^2} C_1^2 \right) + V(\rho) = C_2. \tag{25}$$

This equality allows to find p_1 as a function of ρ:

$$p_1 = \sqrt{2m(C_2 - V(\rho)) - \frac{1}{\rho^2} C_1^2}$$

Then using (19) we end up with the first-order equation for $\rho(t)$ that permit separation of variables

$$m \frac{d\rho}{dt} = \sqrt{2m(C_2 - V(\rho)) - \frac{1}{\rho^2} C_1^2}$$

2.3 Geometrical optics

In geometrical optics, Lagrangian

$$F = w(y)\sqrt{1 + y'^2}$$

corresponds to Euler equation

$$\frac{d}{dt}\left(\frac{w(y)y'}{\sqrt{1+y'^2}}\right) = \frac{dw}{dy}\sqrt{1+y'^2}$$

To find the canonical system, we use the outlined procedure: Define a variable p by the relation $p = \frac{\partial L}{\partial y'}$

$$p = \pm\frac{wy'}{\sqrt{1+(y')^2}}. \tag{26}$$

Solving for y', we obtain first canonical equation:

$$y' = \frac{p}{\sqrt{w^2 - p^2}} = \phi(x, y, p), \tag{27}$$

Excluding y' from expression for $L(x, y, y')$ using (27), we find

$$L(x, y, y') = L_*(x, y, y'(p)) = \frac{w^2}{\sqrt{w^2 - p^2}}.$$

and recalling the representation for the solution y of the Euler equation

$$p' = \frac{\partial L}{\partial y} = \frac{\partial L_*}{\partial w}\,\frac{\partial w}{\partial y}$$

we obtain the second canonical equation:

$$p' = -\frac{w}{\sqrt{w^2 - p^2}}\,\frac{\partial w}{\partial y} \tag{28}$$

Hamiltonian Hamiltonian $H = p\phi - L_*(x, y, p)$ can be simplified to the form

$$H = -\sqrt{w^2 - p^2}$$

It satisfies the remarkably symmetric relation

$$H^2 + p^2 = w^2$$

that contains the whole information about the geometric optic problem. The elegancy of this relation should be compared with messy straightforward calculations that we performed previously.