# Infinitely often oscillating solutions: Vector case

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Figure 1: Convex set, convex function on a convex set

# 1 Convexity: Vector function

## 1.1 Convex function of vector argument

**Convex set, convex hull** A domain $\Omega$ in $R^n$ is called convex if for any points $x_1$ and $x_2$ in $\Omega$ and for any $m$ in the interval $[0,1]$, all points $x = (1-m)\, x_1 + m\, x_2$ belong to $\Omega$. In other words, any point of the line segment belong to $\Omega$ if its ends $x_1$ and $x_2$ are in $\Omega$.

The convex hull or convex envelope $\mathcal{C}\Omega$ of a nonconvex set $\Omega$ is the smallest convex set that contains $\Omega$. It can be also defined as the set of all convex combinations $z$of points $x \in \Omega$.

$$z(x) = \{x: \; x = \sum_i (m_i x_i), \quad \forall x_i \in \Omega \quad \sum_i m_i = 1, m_i \geq 0\}$$

Particularly, the convex envelope of a set of any $n$ points $a_1, \dots a_n$ in $R^n$ is a polygon

$$P(x) = \{x: \; x = \sum_{i=1}^{n} m_i\, a_i, \quad \sum_i m_i = 1, m_i \geq 0\}$$

stretched at these points. Parameters $m_i$ with the stated properties are called the barycentric coordinates of $x$ in the polygone $P$.

**Convex function** Consider a real-valued continuous function $f(x)$, where $x \in R^n$ belongs to a convex set $\Omega$. Function $f$ is called convex if the inequality (**??**) holds, in which $x, v_1, v_2$ are now $n$-vectors not scalars.

Another equivalent geometrical definition of convexity is: $f(x)$ is convex, if the $n+1$-dimensional set $(x, z)$ where $x \in \Omega$ and $z \geq f(x)$ of the points above its graph $y \geq f(x)$ is convex.

**Convexity in a point; Jensen inequality** As in the scalar case, the function $f$ is convex in a point $x$ if

$$f(x) \leq \sum_{i=1}^{n+1} m_i f(x + v_i) \quad \forall m_i, \; v_i \; i = 1, \dots n+1, \quad \text{such that} \tag{1}$$

$$m_i > 0, \quad x + v_i \in \Omega, \quad \sum_{i=1}^{n} m_i = 1, \quad \sum_{i=1}^{n+1} m_i\, v_i = 0 \tag{2}$$

**Derivatives. Hessian** Convex differentiable functions satisfy inequality

$$f(y) \geq f(x) + (y - x)^T \nabla f(x) \quad \forall x, y \in \Omega \tag{3}$$

Second derivatives of a twice differentiable functions is characterized by the Hessian $H(f)$ which is a symmetric $n \times n$ matrix of the second derivatives with entries

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad i, j = 1, \dots n$$

If $f(x)$ is convex, its Hessian is non-negatively defined,

$$z^T H z \geq 0, \quad \forall z \in R^n, \quad |z| \neq 0$$

If $f$ is strictly convex, its Hessian is positively defined.

**Gradient of a convex function is monotone** From (3), one can deduct that monotonicity of a derivative of a convex function, that is an analog of monotonicity of the derivative of a convex function of a scalar argument. We rewrite inequality (3) for the pair $y, x$ instead of $x, y$:

$$f(x) \geq f(y) + (x - y)^T \nabla f(y) \quad \forall x, y \in \Omega$$

and subtract it from (3), obtaining

$$(y - x)^T (\nabla f(y) - \nabla F(x)) \geq 0$$

for all $x, y \in \Omega$. The last inequality is called the monotonicity of a vector-valued function. Monotonicity means that projection of the difference of gradients in any two points to the vector of difference between these points is non-negative. If $f(x)$ is convex, $\nabla f(x)$ is monotone.

**Comment is not clear**

## 1.2 Convex envelope. Vector case

Convex envelope $\mathcal{C}f(x), x \in R^n$ satisfies the equation

$$\mathcal{C}f(x) = \min_{\rho_1, \dots \rho_{n+1}} \sum_{i=1}^{n+1} m_i f(\rho_i), \tag{4}$$

$$x = \sum_{i=1}^{n+1} m_i \rho_i, \quad \sum_{i=1}^{n+1} m_i = 1, \quad m_i \geq 0 \tag{5}$$

that is similar to the scalar case.

**Definition 1.1** The *convex envelope* $\mathcal{C}F$ is a solution to the following minimal problem:

$$\mathcal{C}F(A) = \inf_{v: v + x_0 \in C} \frac{1}{l} \int_C F(A + v(x))\, dx \quad \forall\ v : \int_C v(x)\, dx = 0. \tag{6}$$

This definition determines the convex envelope as the minimum of all parallel secant hyperplanes that intersect the graph of $F$; it is based on Jensen's inequality (**??**).

Figure 2: Supporting points

Figure 3: barycentric coordinates

**Supporting points** To compute the convex envelope $\mathcal{C}F$ one can use the Carathéodory theorem (see [**?**, **?**]). It states that the argument $v(x) = [v_1(x), \dots, v_n(x)]$ that minimizes the right-hand side of (6) takes no more than $n+1$ different values. This theorem refers to the obvious geometrical fact that the convex envelope consists of the supporting hyperplanes to the graph $F(v_1, \dots, v_n)$. Each of these hyperplanes is supported by no more than $(n+1)$ arbitrary points.

The Carathéodory theorem allows us to replace the integral in the right-hand side of the definition of $\mathcal{C}F$ by the sum of $n+1$ terms; the definition (6) becomes:

$$CF(x) = \min_{m_i \in M} \min_{v_i \in v} \left( \sum_{i=1}^{n+1} m_i F(x + v_i) \right), \tag{7}$$

where

$$M = \left\{ m_i : \quad m_i \geq 0, \quad \sum_{i=1}^{n+1} m_i = 1 \right\} \tag{8}$$

and

$$v = \left\{ \ v_i : \quad \sum_{i=1}^{n+1} m_i v_i = 0 \ \right\}. \tag{9}$$

Parameters $m_i$ are called barycentric coordinates of the convex hull stretched at the vertices $x + v_i$.

The convex envelope $\mathcal{C}F(x)$ of a function $F(x)$ at a point $x$ coincides with either the function $F(x)$ or the hyperplane that touches the graph of the function $F$. The hyperplane remains below the graph of $F$ except at the tangent points where they coincide.

The position of the supporting hyperplane generally varies with the point $x$. Fewer than $n+1$ points can support a convex envelope of $F$; in this case, several of the parameters $m_i$ are zero.

On the other hand, the convex envelope is the greatest convex function that does not exceed $F(x)$ in any point $x$ [**?**]:

$$\mathcal{C}F(x) = \max \phi(x) : \ \phi(x) \leq F(x) \ \forall x \quad \text{and } \phi(x) \text{ is convex.} \tag{10}$$

**Example 1.1** Obviously, the convex envelope of a convex function coincides with the function itself, so all $m_i$ but $m_1$ are zero in (7) and $m_1 = 1$; the parameter $v_1$ is zero because of the restriction (9).

The convex envelope of a "two-well" function,

$$\Phi(x) = \min \left\{ F_1(x), \ F_2(x) \right\}, \tag{11}$$

Figure 4: Convex envelope: Cone and paraboloid

where $F_1,\ F_2$ are convex functions of $x$, either coincides with one of the functions $F_1,\ F_2$ or is supported by no more than two points for every $x$; supporting points belong to different wells. In this case, formulas (7)–(9) for the convex envelope are reduced to

$$\mathcal{C}\Phi(x) = \min_{m,v} \{mF_1(x-(1-m)v) + (1-m)F_2(x+mv)\}\,. \qquad (12)$$

Indeed, the convex envelope touches the graphs of the convex functions $F_1$ and $F_2$ in no more than one point. Call the coordinates of the touching points $x+v_1$ and $x+v_2$, respectively. The restrictions (9) become $m_1v_1 + m_2v_2 = 0,\ m_1 + m_2 = 1$. It implies the representations $v_1 = -(1-m)v$ and $v_2 = mv$.

**Example 1.2** Consider the special case of the two-well function,

$$F(v_1, v_2) = \begin{cases} 0 & \text{if} \quad v_1^2 + v_2^2 = 0, \\ 1 + v_1^2 + v_2^2 & \text{if} \quad v_1^2 + v_2^2 \neq 0. \end{cases} \qquad (13)$$

The graph of function $F(v_1, v_2)$ is axisymmetric in the plane $v_1,\ v_2$; therefore, the convex envelope is axisymmetric as well: $\mathcal{C}F(v_1, v_2) = f(\sqrt{v_1^2 + v_2^2})$. It is therefore enough to construct the envelope of function $F(v)$, where $v = \sqrt{v_1^2 + v_2^2}$

$$F(v) = \begin{cases} 0 & \text{if} \quad v = 0, \\ 1 + v^2 & \text{if} \quad v^2 \neq 0. \end{cases} \qquad (14)$$

The convex envelope $\mathcal{C}F(v)$ is supported by the point $v_a = 0$ and by a point $v_b$ that (i) belongs to the parabola $f(v) = 1 + v_b^2$ and (ii) is such that the tangent line to the parabola at the point $v_b$ passes through the origin. The equation of the tangent line in the plane $v, y$ is $y - f(v_b) = f'(b)(v - v_b)$. Setting $y = v = 0$ due to (ii), we find $f(v_b) = f'(v_b)v_b$ or $1 + v_b^2 = 2v_b^2$ and $v_b = 1$. The values of $F$ are: $F(v_1) = 0,\ F(V_b) = 2$, and the equation for the envelope is

$$\mathcal{C}F(v) = \begin{cases} 2v & 0 \leq v \leq 1 \\ 1 + v^2 & 1 \leq v \end{cases}$$

. Coming back to original notations we find the supporting circumferences of $F(v_1, v_2)$:

$$A: \ (v_1, v_2) = (0, 0), \quad B: \ (v_1, v_2): \ v_1^2 + v_2^2 = 1$$

and the surface of the envelope is

$$\mathcal{C}F(v_1, v_2) = \begin{cases} 2\sqrt{v_1^2 + v_2^2} & \text{if} \quad v_1^2 + v_2^2 \leq 1, \\ 1 + v_1^2 + v_2^2 & \text{if} \quad v_1^2 + v_2^2 > 1. \end{cases} \qquad (15)$$

The envelope is a cone if it does not coincide with $F$, $\mathcal{C}F < F$, and a paraboloid if it coincides with $F$, $\mathcal{C}F = F$.

Figure 5: Three-well function

**Hessian of Convex Envelope** We mention here property of the convex envelope that we will use later. If the convex envelope $\mathcal{C}f(x)$ does not coincide with $f(x)$ for some $x = x_0$, then $\mathcal{C}F(x_0)$ is convex, but not strongly convex. At these points the Hessian $H((f)$ is semipositive; its determinant is zero:

$$H(\mathcal{C}f(x)) \geq 0, \quad \det H(\mathcal{C}f(x)) = 0 \quad \text{if } \mathcal{C}f < f \tag{16}$$

which say that $H(\mathcal{C}f)$ is a nonnegative degenerate matrix. These relations can be used to compute $\mathcal{C}f(x)$.

For example, compute the Hessian $H$ of the cone $F(v_1, v_2) = 2\sqrt{v_1^2 + v_2^2}$, from (15). We have

$$H = \frac{1}{(v_1^2 + v_2^2)^{\frac{3}{2}}} \begin{pmatrix} v_2^2 & -v_1 v_2 \\ -v_1 v_2 & v_1^2 \end{pmatrix}$$

and we see that $\det(H) = 0$.

## 1.3 Convex envelope of a three-well function

The convex envelope is a multi-face surface. The next problem demonstrates the variety of the components of its surface.

Describe convex envelope $\mathcal{C}f$ of three-well function $f(x_1, x_2)$

$$f(x_1, x_2) = \min\{\phi_1, \phi_2, \phi_3\} \tag{17}$$
$$\phi_1 = x_1^2 + x_2^2 \tag{18}$$
$$\phi_2 = x_1^2 + (x_2 - 1)^2 \tag{19}$$
$$\phi_3 = (x_1 - 1)^2 + x_2^2 \tag{20}$$

Convex functions $\phi_i$ are called wells.

The convex envelope is a multi-face surface that is stretched between the wells. No more than three supporting points support each component of the envelope; the convex wells contain no more than one supporting point each.

The convex envelope is a solution to the optimization problem

$$\mathcal{C}f(x) = \min_m \min_\rho \sum_{i=1}^{3} m_i \phi_i(\rho_i) \tag{21}$$
$$x = m_1\rho_1 + m_2\rho_2 + m_3\rho_3, \tag{22}$$
$$m_1 + m_2 + m_3 = 1, \quad m_i \geq 0, \quad i = 1, 2, 3. \tag{23}$$

Here, $m_i$ are barycentric coordinates of $x$ in the triangle with vertices at $\rho_i$.

Convex envelope $\mathcal{C}f$ consists of several components:

**Bottom component** The bottom part $\Omega_0$ is correspond to the case when all $m_i > 0$; the minimization with respect to $\rho_i$ gives: =

$$\rho_1 = (0,0), \quad \rho_2 = (1,0), \quad \rho_3 = (0,1)$$

The envelope is supported by three points $\rho_i$ in three wells. Argument $x$ belongs to a convex hull $\Omega_0$, stretched on these points $x \in \Omega_0$,

$$\Omega_0 = \{x_1, x_2 : \ (x_1, x_1, x_2) = \sum_{i=1}^{3} \mu_i \rho_i, \quad \sum_{i=1}^{3} \mu_i = 1, \quad \mu_i \geq 0\}$$

We compute:

$$\Omega_0 = \{x_1, x_2 : \ x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}, \tag{24}$$

The values of $\phi_i$ are, respectively:

$$\phi_1(\rho_1) = 0, \quad \phi_2(\rho_2) = 0, \quad \phi_3(\rho_3) = 0$$

The convex envelope in $\Omega_0$ is

$$\mathcal{C}f(x_1, x_2) = \sum_{i=1}^{3} \mu_i \phi_i(\rho_i) = 0 \quad \text{if} \quad (x_1, x_2) \in \Omega_0, \tag{25}$$

The coordinates of a point in the convex hull are

$$x_1 = \mu_2, \quad x_2 = \mu_3, \quad 0 \leq \mu_2 + \mu_3 \leq 1, \quad \mu_2 \geq 0, \quad \mu_3 \geq 0$$

Notice that supporting points do not vary with $x \in \Omega_0$, only the barycentric coefficients $\mu_i$ ($\mu_3 = 1 - \mu_1 - \mu_2$) vary.

**Side components** First side component of the boundary corresponds to the case when $m_3 = 0$. This component is supported by by two points at two two convex wells $\phi_1$ and $\phi_2$. The domain $\Omega_1$ that support this case, is

$$\Omega_1 = \{x_1, x_2 : \ (x_1, x_2) = \mu_1(0, x_2) + \mu_2(1, x_2), \quad m_1 + \mu_2 = 1, \quad \mu_i \geq 0$$

it is a strip:

$$\Omega_1 = \{x_1, x_2 : \ x_1 \in [0,1], \quad x_2 \in [-\infty, 0], \tag{26}$$

The supporting points are

$$\rho_1 = (0, x_2), \quad \rho_2 = (1, x_2)$$

We compute

$$\phi_1(0, x_2) = x_2^2, \quad \phi_1(1, x_2) = x_2^2,$$

The convex envelope in the region $\Omega_2$ is

$$\mathcal{C}f(x_1, x_2) = \mu_1 \phi_1(0, x_2) + \mu_2 \phi_2(1, x_2) = x_2^2, \quad (x_1, x_2) \in \Omega_1, \tag{27}$$

Figure 6: Contourplot of the convex envelope

(Here, the coordinate $x_1$ is $x_1 = \mu_2$ and $x_1 \in (0,1)$. This part lies between two convex wells $\phi_1$ and $\phi_2$ and consists of moving parallel intervals supported by two points at these wells. This type of surface is called a ruled surface, that is a surface that can be swept out by moving a line in space. A variation of position $x \in \Omega_2$ along the direction $x_1$ results in the variation of $\mu_1 = 1 - \mu_2$ with a fixed position of the supporting points, and a variation along the direction $x_2$ results in the variation of supporting points $\rho_1$ and $\rho_2$ with a fixed fraction $m_2$.

The second side component of the envelope correspond to $m_2 = 0$ and $m_1, m_3 > 0$. This part is similar to the previous case, it is obtained from it by interchanging indices. We have

$$\Omega_2 = \{x_1, x_2 : \ x_2 \in [0,1], \quad x_1 \in [-\infty, 0]\} \tag{28}$$

$$\rho_1 = (x_1, 0), \quad \rho_3 = (x_1, 1)$$

$$\mathcal{C}f(x_1, x_2) = x_1^2, \quad (x_1, x_2) \in \Omega_2 \tag{29}$$

The third component correspond to $m_1 = 0$ and $m_2, m_3 > 0$. Similarly to the previous case we compute,:

$$\Omega_3 = \{x_1, x_2 : \ |x_1 - x_2| \in [0,1], \quad x_1 + x_2 \in [1, \infty]\} \tag{30}$$

$$\rho_2 = (x_1, 0), \quad \rho_3 = (x_1, 1)$$

$$\mathcal{C}f(x_1, x_2) = (x_1 + x_2)^2, \quad (x_1, x_2) \in \Omega_3 \tag{31}$$

**Regions of convexity** The remaining three regions correspond to the case when one of coordinates $m_i$ equals to one, and the other two are equal to zero. In these cases, the convex envelope coincides with the function itself, $f(x)$ is convex in these regions.

We compute

$$\mathcal{C}f = \phi_1, \quad \text{in } \Omega_4 = \{(x_1, x_2) : \ x_1 \leq 0, x_2 \leq 0 \tag{32}$$

$$\mathcal{C}f = \phi_2, \quad \text{in } \Omega_5 = \{(x_1, x_2) : x_2 > 1, \ 1 \geq x_2 - x_1 \tag{33}$$

$$\mathcal{C}f = \phi_3, \quad \text{in } \Omega_6 = \{(x_1, x_2) : x_1 > 1, \ 1 \geq x_1 - x_2 \tag{34}$$

In Figure 6 the contour plot of the obtained convex envelope is shown.

# 2 Relaxation of problems with vector minimizer

## 2.1 Relaxation procedure

The procedure is essentially the same. A bounded from below of Lagrangian $F(x, u, v)$ of superlinear with respect of $z$ growth with a the non-convex with respect of $v$ region is replaced with its convex envelope $F_v(x, u, v)$ . Every point of the convex envelope of a function of $n$-dimensional vector is a convex combination by $n+1$ supporting points $\rho_1, \rho_{n+1}$,

$$v = \sum_{i=1}^{n+1} m_i \rho_i, \quad \sum_{i=1}^{n+1} m_i = 1, \quad m_i \geq 0$$

The minimizing sequence is a fast oscillating vector function $v(x)$ takes not more that $n+1$ values in each infinitesimal interval. The relaxed Lagrangian is

$$RF(x, u, u') = \min_{m_1 \dots m_{n+1}} \left( \min_{\rho_1 \dots \rho_{n+1}} \sum_{i=1}^{n+1} m_i F(x, u, u') \right) \tag{35}$$

$$u' = \sum_{i=1}^{n+1} m_i \rho_i, \quad \sum_{i=1}^{n+1} m_i = 1, \quad m_i \geq 0 \tag{36}$$

Calculating the minima, we express the relaxed Lagrangian through convex envelope with respect to $v = u'$

$$RF(x, u, u') = \mathcal{C}F(x, u_h, u_h') \tag{37}$$

Here $u_h$ is the homogenized (averaged over a small $\epsilon$-interval) minimizer, $u_h'$ is its homogenized derivative.

Instead of one minimizer $u(x)$ in the original problem with nonconvex Lagrangian, the relaxed problem depends on several minimizers: the supporting points $\rho_i(x)$ of the convex envelope and the barycentric coordinates $m_i(x)$. These continuous functions describe parameters of infinitely often oscillating minimizing sequence with the derivative that sequentially takes values $\rho_1, \dots \rho_n$ in the infinitely small intervals $[x, x+\epsilon]$.

**Remark 2.1** Analyzing the homogenized solution, one cannot determine what value takes $u'$ at a specific point $x$ but only the relative length (measure) of the intervals where a specific value is taken. Such fast oscillating sequences are called solutions in Young measures.

## 2.2 Examples of nonconvex problems for vector minimizer

**Three-well Lagrangian** Consider the problem with the Lagrangian

$$\mathcal{C}F(v_1, v_2) + \Phi(x, u_1, u_2)$$

where $\mathcal{C}F$ is the convex envelope of three-well function described in example 37.

In the domains where the Lagrangian is convex and the convex envelope $\mathcal{C}F(v_1, v_2)$ coincides with the wells in $F(v_1, v_2)$, stationarity conditions are represented by a system of two second-order Euler equations:

$$2u_1'' - \frac{\partial \Phi}{\partial u_1} = 0, \quad 2u_2'' - \frac{\partial \Phi}{\partial u_2} = 0,$$

Notice that linear with respect to derivative terms in the second and third wells are null-Lagrangians and they do not affect Euler equations, because

$$\frac{d}{dx}\left(\frac{\partial(u'-1)^2}{\partial u'}\right) = 2\frac{d(u'-1)}{dx} = 2u''$$

In the complex hull $\Omega_0$, where

$$\mathcal{C}F(v_1, v_2) = 0$$

stationarity is described by algebraic equations for $u_1$ and $u_2$:

$$\frac{\partial \Phi}{\partial u_1} = 0, \quad \frac{\partial \Phi}{\partial u_2} = 0, \tag{38}$$

These minimizers are zigzag functions which derivatives taken pointwize values $\rho_1 = (0,0), \rho_2 = (1,0)$, and $\rho_3 = (0,1)$. The weights (measures) $m_i$ are found by differentiation of the conditions (38) and (36):

$$\frac{d}{dx}\frac{\partial \Phi}{\partial u_1} = \frac{\partial^2 \Phi}{\partial x \partial u_2} + \frac{\partial^2 \Phi}{\partial u_1^2}u_1' + \frac{\partial^2 \Phi}{\partial u_1 \partial u_2}u_2' \tag{39}$$

$$\frac{d}{dx}\frac{\partial \Phi}{\partial u_2} = \frac{\partial^2 \Phi}{\partial x \partial u_2} + \frac{\partial^2 \Phi}{\partial u_1 \partial u_2}u_1' + \frac{\partial^2 \Phi}{\partial u_2^2}u_2' \tag{40}$$

This two equations are linear relations for $m_1, m_2, m_3$ because

$$u' = \sum_{i=1}^{3} m_i \rho_i;$$

together with the third equation $m_1 + m_2 + m_3 = 1$, they allow for finding barycentric coordinates $m_i$.

In the remaining domains, stationarity conditions include one second-order differential equation and one algebraic equation. For example, in the domain $\Omega_1$, the relaxed Lagrangian is

$$F_2 = (u_2')^2 + \Phi,(x, u_1, u_2), \quad u_1' \in [0,1]$$

the Euler equations are

$$\frac{\partial \Phi}{\partial u_1} = 0 \quad 2u_2'' = \frac{\partial \Phi}{\partial u_2} = 0$$

Barycentric coordinates $m_1, m_2$, $m_1 + m_2 = 1$, are found by differentiation of the first stationarity equation as in (40)

$$\frac{d}{dx}\frac{\partial \Phi}{\partial u_1} = \frac{\partial^2 \Phi}{\partial x \partial u_1} + \frac{\partial^2 \Phi}{\partial u_1^2}u_1' = 0$$

Express $u_1'$ as function of $m_1$: $u_1' = m_1\rho_1 + m_2\rho_2$ where $Gr_1 = 0, Gr_2 = 1$ and $m_2 = 1 - m_1$: $u_1' = m_2$ and the stationarity conditions, we find

$$m_2 = -\frac{\partial^2\Phi}{\partial x \partial u_1}\left(\frac{\partial^2\Phi}{\partial u_1^2}\right)^{-1}$$

The other two cases are treated similarly.

We can also check that determinant of the Hessian is zero everywhere, where $\mathcal{C}F < F$.

## 2.3 Conclusion and Problems

We have observed the following:

- A variational problem has the fine-scale oscillatory minimizer if its Lagrangian $F(x, u, u')$ is a nonconvex function of its third argument.
- Homogenization leads to the relaxed form of the problem that has a classical solution and preserves the cost of the original problem.
- The relaxed problem is obtained by replacing the Lagrangian of the initial problem by its convex envelope. It can be computed as the second conjugate to $F$.
- The dependence of the Lagrangian on its third argument in the region of nonconvexity does not affect the relaxed problem.

To relax a variational problem, we use two ideas. First, we replaced the function with its convex envelope and got a stable extension of the problem. Second, we proved that the value of the integral of the convex envelope $\mathcal{C}F(\boldsymbol{v})$ of a given function is equal to the value of the integral of this function $F(\boldsymbol{v})$ if its argument $\boldsymbol{v}$ is a zigzag curve. We use the Carathéodory theorem, which tells that the number of subregions .whe constancy of the argument is less than or equal to $n + 1$, where $n$ is the dimension of the minimizer.

**Regularization and relaxation** The considered nonconvex problem is another example of an ill-posed variational problem. For these problems, the classical variational technique based on the Euler equation fails to work. Here, The limiting curve is not a discontinuous curve as in the previous example, but a limit of infinitely fast oscillating functions, similar to $\lim_{\omega\to\infty}\sin(\omega x)$.

We may apply regularization to discourage the solution to oscillate. Doing this, we pass to the problem

$$\min_u \int_0^1 (\epsilon^2(u'')^2 + G(u, u'))dx$$

that corresponds to Euler equation:

$$\begin{array}{ll} \epsilon^2 u^{IV} - u'' + u = 0 & \text{if} \quad |u'| \geq \frac{1}{2} \\ \epsilon^2 u^{IV} + u'' + u = 0 & \text{if} \quad |u'| \leq \frac{1}{2}. \end{array} \tag{41}$$

The Weierstrass condition this time requires the convexity of the dependence of Lagrangian on $u''$; this condition is satisfies.

The solution of Euler equations is oscillatory, with the period of oscillation of the order of $\epsilon$. It $\epsilon \to 0$, the solution still tends to an infinitely often oscillating distribution. When $\epsilon$ is positive but small, the solution has a finite but large number of wiggles. The computation of such solutions is difficult and some times unnecessary: It strongly depends on an artificial parameter $\epsilon$, which is difficult to justify physically. It is n=more natural to replace an ill-posed problem with a *relaxed* one. The idea of relaxation is in a sense opposite to the regularization. Instead of discouraging fast oscillations, we admit them as legitimate minimizers and describe such minimizers in terms of smooth functions: the limits of oscillating variable and the average time that it spends on each boundary.