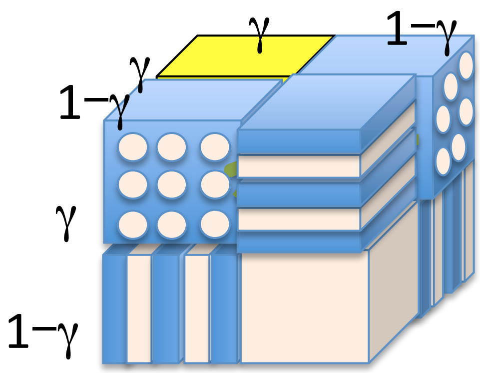


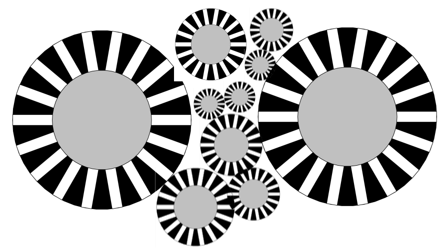

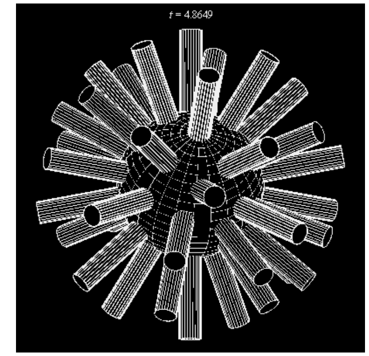

ICIAM15 Beijing, China

# Amazing Structures of Optimal Multicomponent Composites

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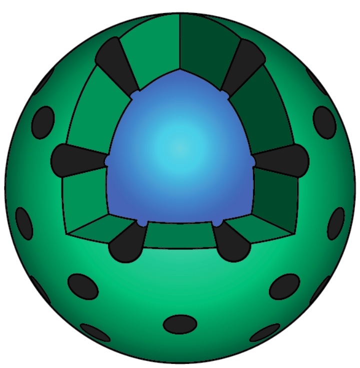
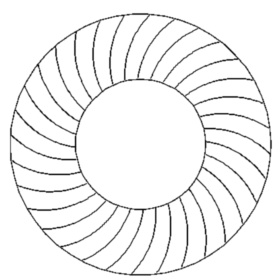
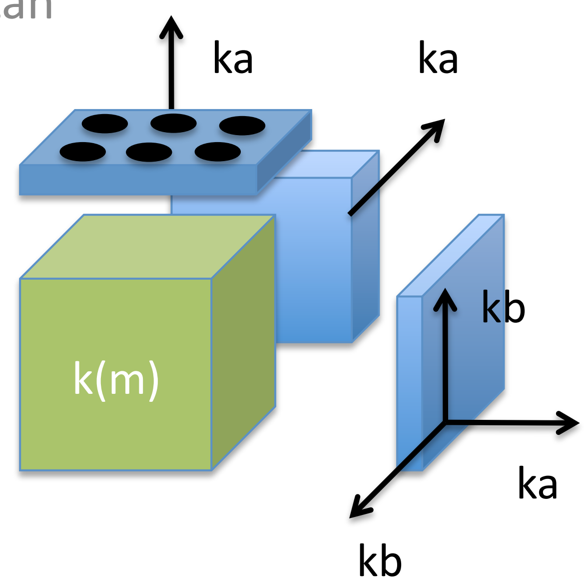
supported by DMS NSF







# Optimal structures

- Structures, that minimize the sum of energies in several homogeneous and orthogonal external fields, are called *"optimal"*.
  - ✓ P1: Minimal energy in a fixed current - minimal total resistivity,
  - ✓ P2: Minimal energy in a fixed electrical field - minimal conductivity.

**Optimal structures = minimizing sequences in the quasiconvex envelope problem.**

$$J = \inf_{\chi} W; \quad W = \inf_{v} \int_{\Omega} \left[ v^T K_i v + \gamma_i \right] \chi_i \, dx, \quad v \text{ is } \Omega\text{-periodic}, \quad \int_{\Omega} v \, dx = v_0$$

$$\sum_{j,k} A_{ijk} \frac{\partial v_j}{\partial x_k} = 0, i = 1,..p \quad (\nabla \cdot v = 0, \nabla \times v = 0)$$

$$\chi_i = \begin{cases} 1 \text{ if } x \in \Omega_i \\ 0 \text{ if } x \notin \Omega_i \end{cases}, \quad \Omega = \bigcup \Omega_i .$$

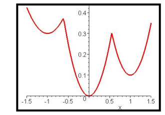
Instead of the cost $\gamma_{\mathrm{i}}$ we fix volume fractions $m_i = \int_{\Omega} \chi_{\mathrm{i}}$

# Motivations

- Technological capabilities, such as 3d printing and microfabrication, allow a huge variety of structures to be manufactured for roughly the same price, and one wants to know ``the best'' structure.

- Mathematically, the best composite structure is a solution to the quasiconvex envelope problem for a multiwell Lagrangian. Few examples for two-well Lagrangians are known (starting with Lurie &Ch 82, 84, see also Tartar 85) but the technique for multiwell quasiconvex envelopes is in its infancy.

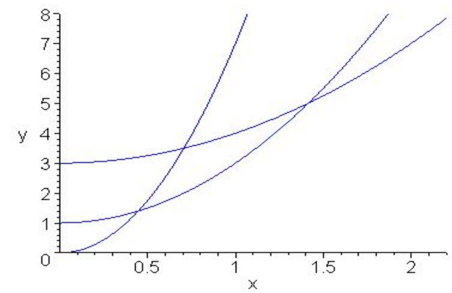
- Aesthetically, the optimal structures are delightful, and intellectually, they present an elegant riddle that links geometry and PDE



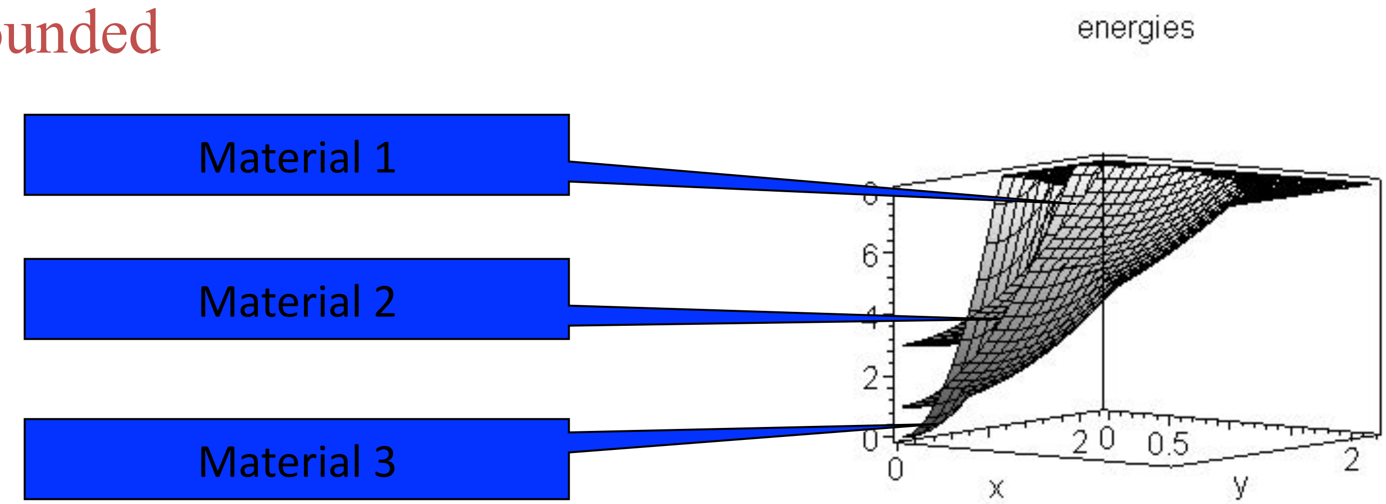
# Comment on minimizers

Contrary to homogenization, the problem of optimal composite asks for a partition (may be a fractal-type one) of the domain, that corresponds to extremal average properties.

Materials (wells) are identified by the range fields (gradients).

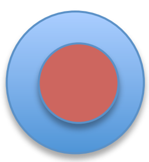
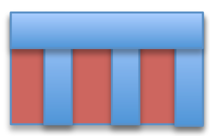



## Fields in materials are ordered and bounded



# Features

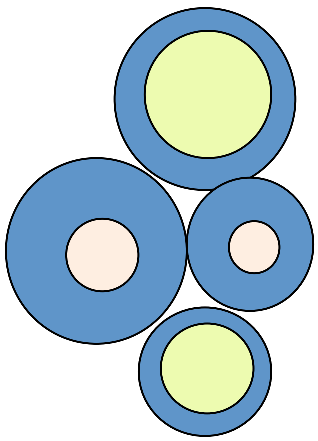
- **Simple fields - complex structures** In optimal structures, the fields *v(x)* or some their components are constant in Omega_i. To provide the constancy, the structures typically become fractals.
- The optimality is established by **geometrically independent bounds** and corresponding sufficient optimality conditions on *v(x)*. The bounds account for corollaries of differential properties of *v(x)* . They might be exact or rough. Exact bounds are realized by optimal structures (or a sequence of structures)
- The geometry of two-material optimal composites is steady and intuitive,

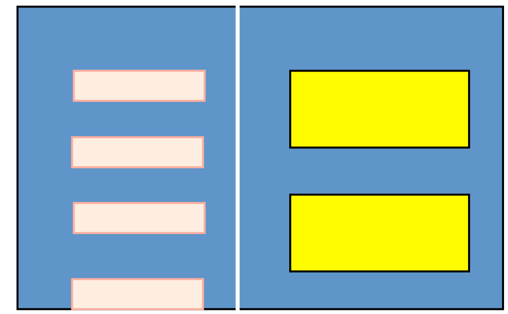



but geometry of three-material optimal composites is complex, typically non–unique, and cannot be simply guessed.

# First optimal three-phase structures
# Require sufficient large volume fractions of the "best" material k_1.

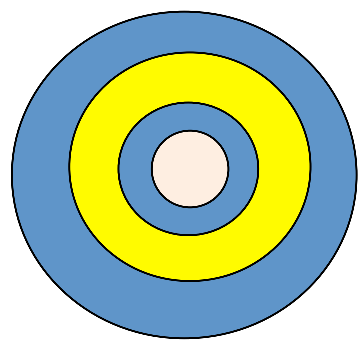
k1-k3 coated circles mimic  k2
and become "invisible"

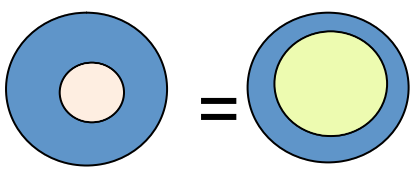


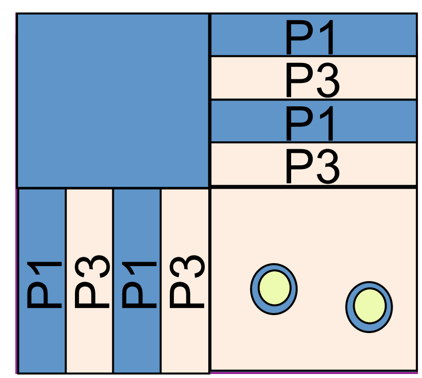



Multicoated spheres
Lurie, Ch 86

**Parallel matrix laminates**
**Milton Kohn, 88**








Attainability condition

$$\mathbf{k}_* - k_2\mathbf{I} \leq 0$$

Parallel coated spheres
Milton 82

Sigmund, Gibiansky, 2000.

# Beyond translation method

- Translation method accounts for differential constraints through an analog of Green’s theorem and provides translation bounds.
  - Translation bound is exact for two-well problem and for the case when the volume fraction m1  of the “best material” is larger than a threshold.
  - Translation bound is definitely loose when m1 -> 0.
- One assumes that the fields in an optimal geometry are subject to some additional inequalities that become exact for small m1.

- **Question: how to reveal these inequalities?**

# How the inequalities were revealed for 2d problem

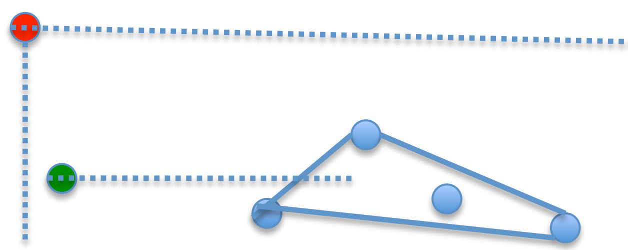
- Gibiansky and Sigmund (2002) suggested a new class of isotropic optimal structures
- Albin, Ch, Nesi investigated the fields in these and similar structures and extended results to anisotropic composites.
- Ch (2009) investigated fields in materials in limiting structures (minimal values of m1). conjectured and proved new inequalities for optimal fields; in the asymptotical case, they coincide with previously found inequalities by Alessandrino and Nesi.
- When these inequalities are added to the Translation method procedure, the novel bounds emerge, that are achievable everywhere.
- For anisotropic bounds, additional inequalities were conjectured by Ch. and G.Dzierzanowski; the proofs are no be perfected.

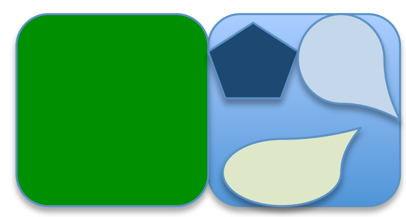
# Meaning of the inequalities

$$A: \quad \det(e_1 - e_3) \geq 0$$

follows from compatibility of the supporting points of the envelope:

Any supporting point is in a rank-one connection with a point in the convex envelope of other supporting points

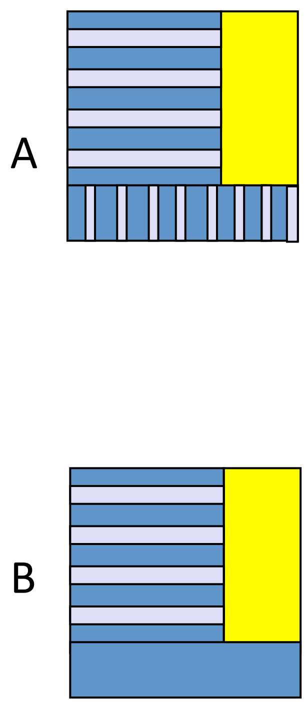


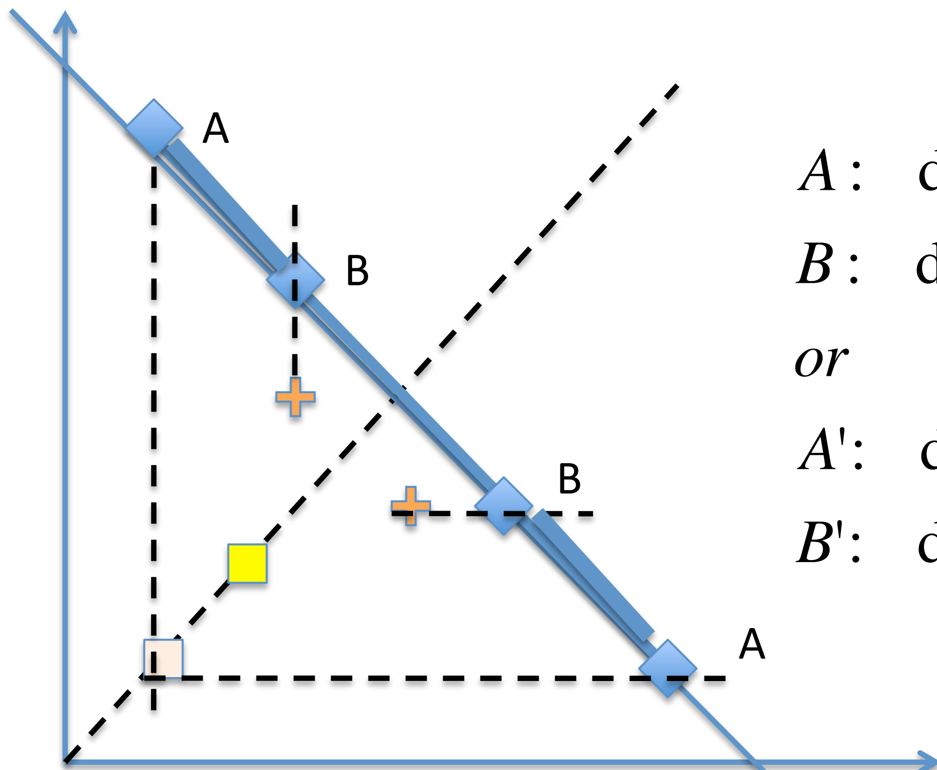


$$B: \quad \det(e_1 - e_0) \leq 0$$

is an optimization requirement (work in progress). The optimality test disproves the field in the forbidden interval

# Predicted location of optimal fields in the "best" material




$A: \quad \det(e_1 - e_3) \geq 0$

$B: \quad \det(e_1 - e_0) \leq 0$

*or*

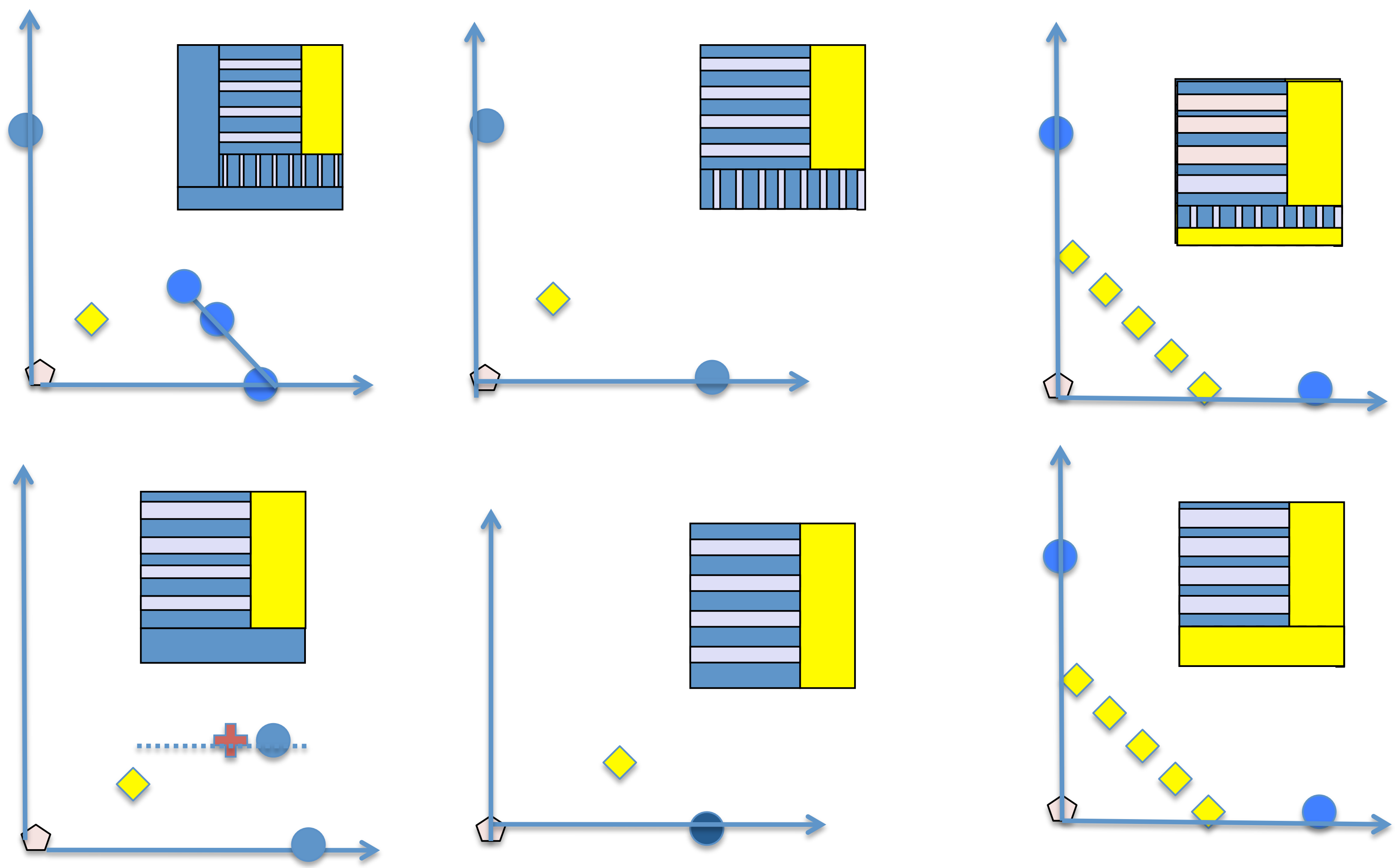
$A': \quad \det(k_1 e_1 - k_3 e_3) \geq 0$

$B': \quad \det(k_1 e_1 - k_0 e_0) \leq 0$

When $m_1 \to 0, \quad \det(e_1 - e_3) = 0$

When anisotropy incereases, $e_1$ oscillates between $\det(e_1 - e_3) = 0$ and $\det(e_1 - e_0) = 0$

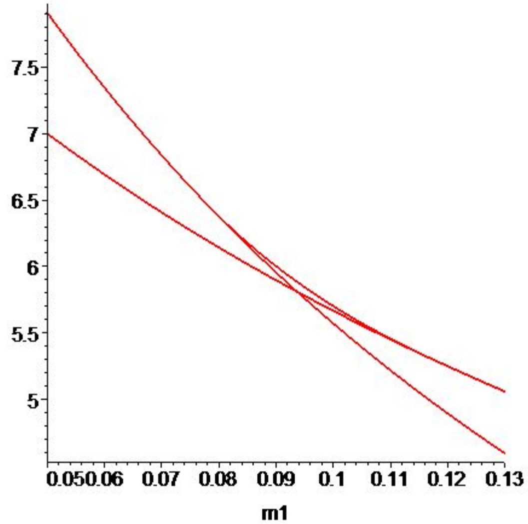
# Eigenvalues of supports of quasiconvex envelope (fields in optimal materials), required by the bounds and matching optimal structures



# Example: Isotropic Bounds, k3=infinity

Th.

(a) Effective conductivity $k_*$ of every isotropic three-phase composite satisfies the bound (Special case of Nesi (1995) and Ch. (2009) bounds)




$$k_* \geq \begin{cases} -k_1 + \left( \dfrac{m_1}{2k_1} + \dfrac{m_2}{k_1 + k_2} \right)^{-1} & \text{if } m_1 \geq m_{11} \\ k_2 + 2\dfrac{k_1}{m_1}\left(1 - \sqrt{m_2}\right)^2 & \text{if } m_{12} \leq m_1 \leq m_{11} \\ -k_2 + \left( \dfrac{m_1}{2k_1} + \dfrac{m_2}{2k_2} \right)^{-1} & \text{if } m_1 \leq m_{12} \end{cases}$$

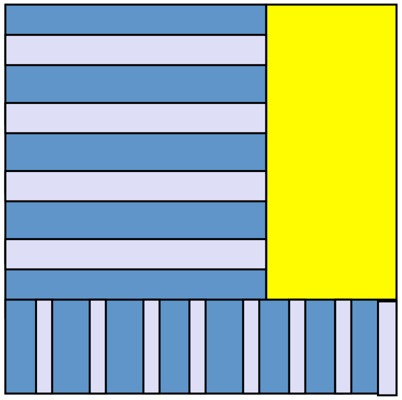
$$m_{11} = 2\frac{k_1}{k_1 + k_2}\left(\sqrt{m_2} - m_2\right), \quad m_{12} = \frac{k_1}{k_2}\left(\sqrt{m_2} - m_2\right)$$

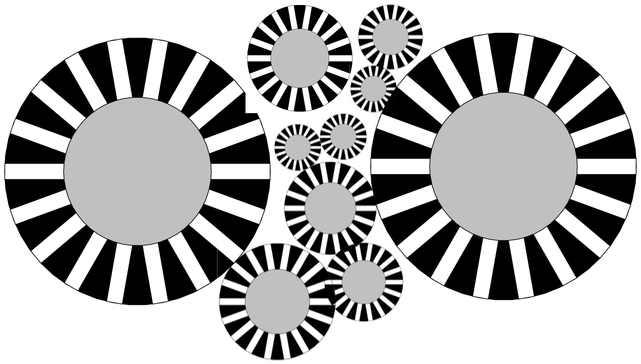
(b) The bound is exact everywhere.

# Optimal structures

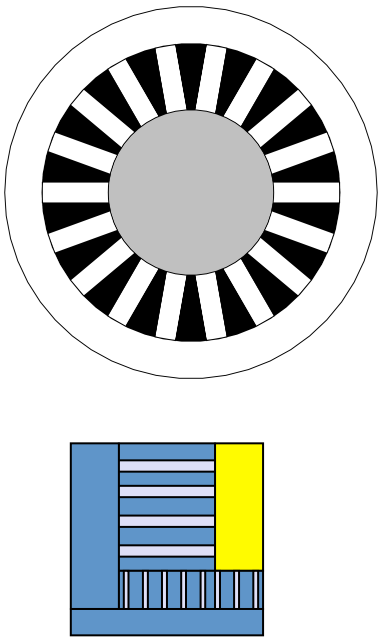
1. Repeated laminates: Use laminate formulae repeatedly in different scales.
2. Coated-spheres-type wheels assembly (effective field theory)

The assembly elements, “wheels”, vary depending on the fractions of materials.

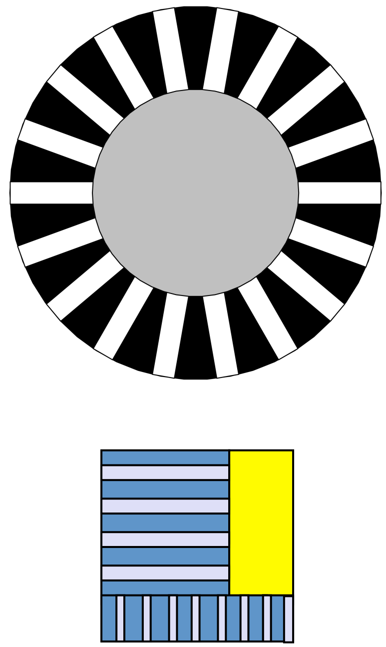




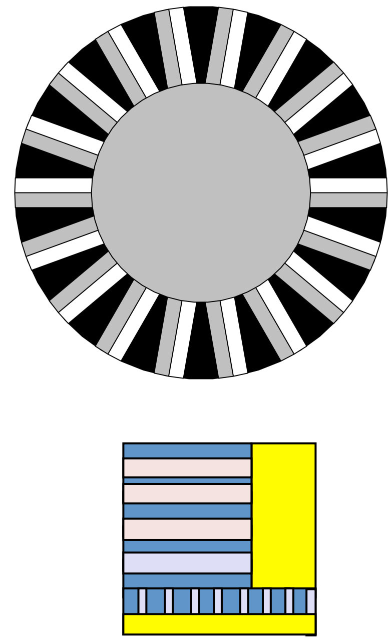
# Assembly elements: the optimal topology depends on $m_1$



$m_1>m_{11}$
Hashin-Shtrikman bound. $K_1$ is connected



$m_{12}<m_1<m_{11}$
No connected phases



$m_1<m_{12}$
$k_2$ (gray) is connected

## 3D problem: search for optimal structures

Problem: so far, the only known bound for 3d is the translation (Hashin–Shtrikman) bound.

The optimal structures should hint for new inequalities and new bounds. We want to extend the domain of applicability for the known bounds, then investigate the relations on the fields in materials in the limiting structures, conjunct and (hopefully) prove new inequalities and derive the bounds.

Difficulties: Laminates need a large number of iterations, choice of substructures, and other not obvious decisions.

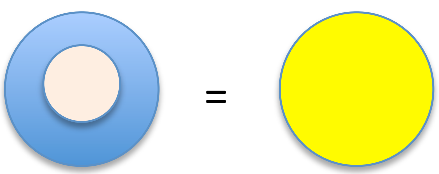
# P1. Optimal structures of minimal resistance

Given three materials with conductivities k1, k2, and k3:
k1>k2 > k3=0  (k3 is a perfect insulator)
Find a range of  isotropic structures of maximum effective conductivity.

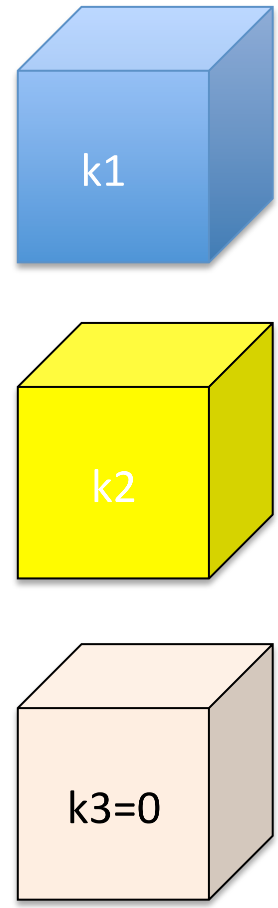
$$v_0^T k_0 v_0 + \gamma_0 = \inf \int_\Omega F(v)\,dx, \quad \nabla \times v = 0,$$

$$F = \min\left(v^T k_1 v + \gamma_1, v^T k_2 v + \gamma_2, v^T k_3 v\right)$$

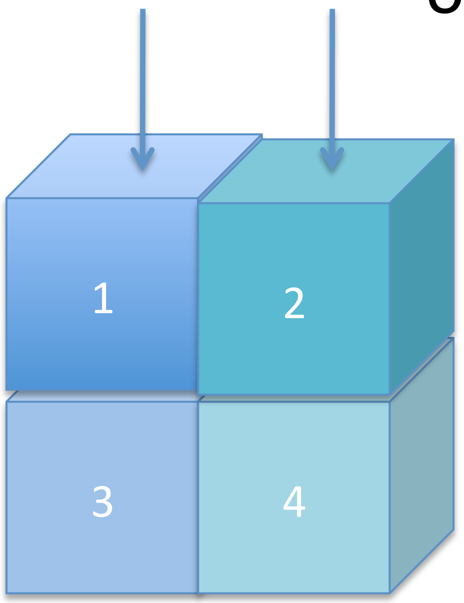
Known structures: Range of realizability of H-S structures by Coated spheres structures:



$$m_1 \geq m_1^{cr} = \frac{3k_2}{k_2 + 2k_1}(1 - m_2)$$



# Columnar structure: Condition of piece-wise constancy of fields in bulk structures




After Zhikov, Sigmund.. – separation of variables

$$k_1 e_1 = k_3 e_3, \quad k_2 e_2 = k_4 e_4$$

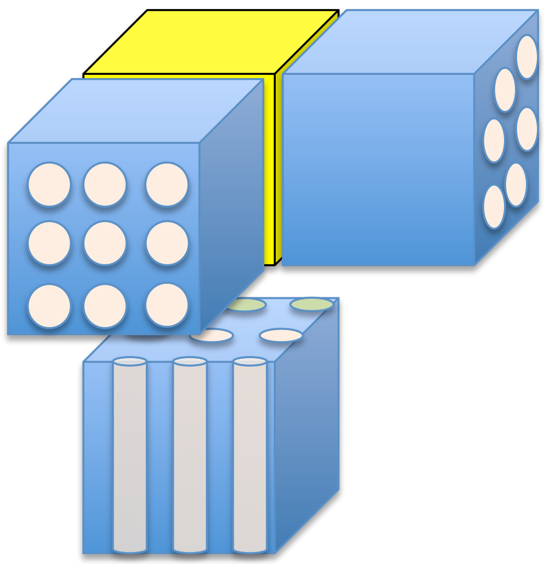
$$\text{if } \frac{k_1}{k_3} = \frac{k_2}{k_4} \text{ then } e_i \text{ are constant } i = 1..4$$

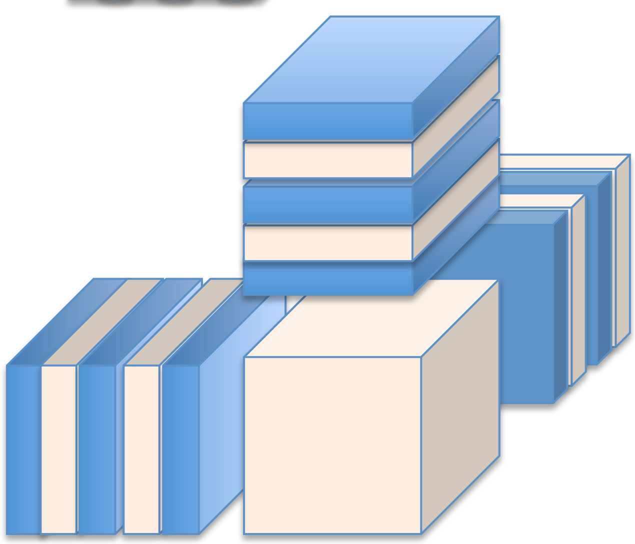
Effective properties:

$$\text{If } \frac{k_1}{k_3} = \frac{k_2}{k_4} \text{ then}$$

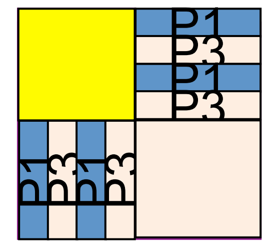
$$k_0 = S_A \frac{k_1 k_3}{\gamma k_3 + (1-\gamma) k_1} + S_B \frac{k_2 k_4}{\gamma k_4 + (1-\gamma) k_2}$$

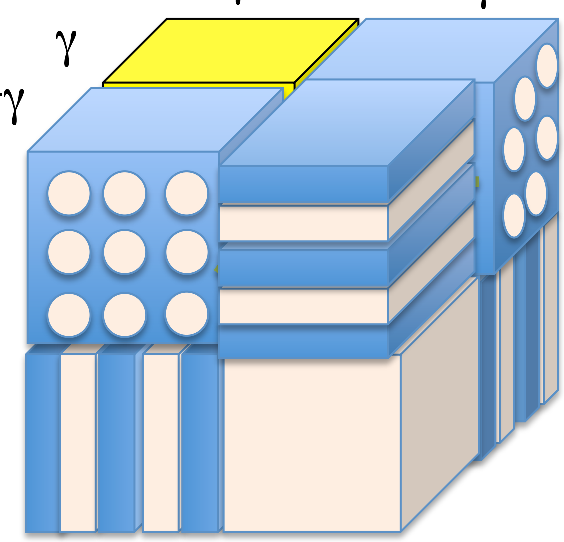
# Bulk blocks assemblage - 1





Inspired by
Sigmund-
Gibiansky
structures,
studied by N.Albin for his PhD.

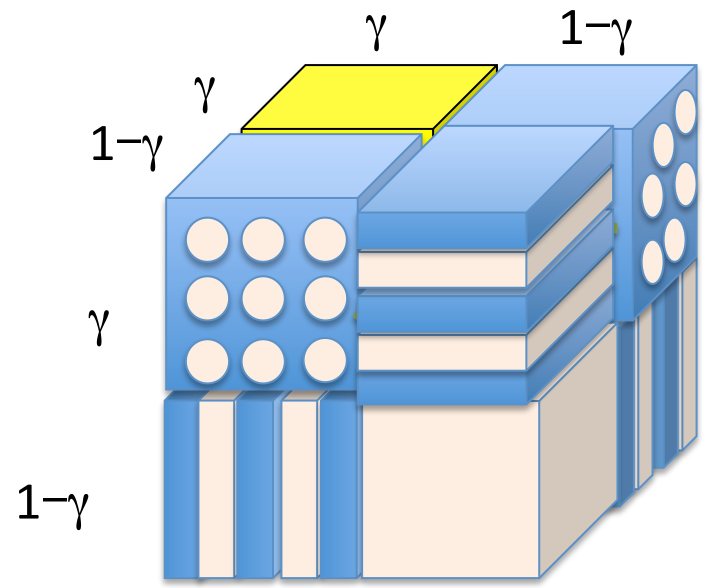


# Parameters

- The volume fractions *cL* and *c* of *k1* in the laminates and cylinders, respectivally,  are connected to satisfy the columnar condition.

$$c_L = \frac{k_1 c^2}{k_2(2-c)},$$




- The volume fractions and effective conductivity are expressed through structural parameters parameters:

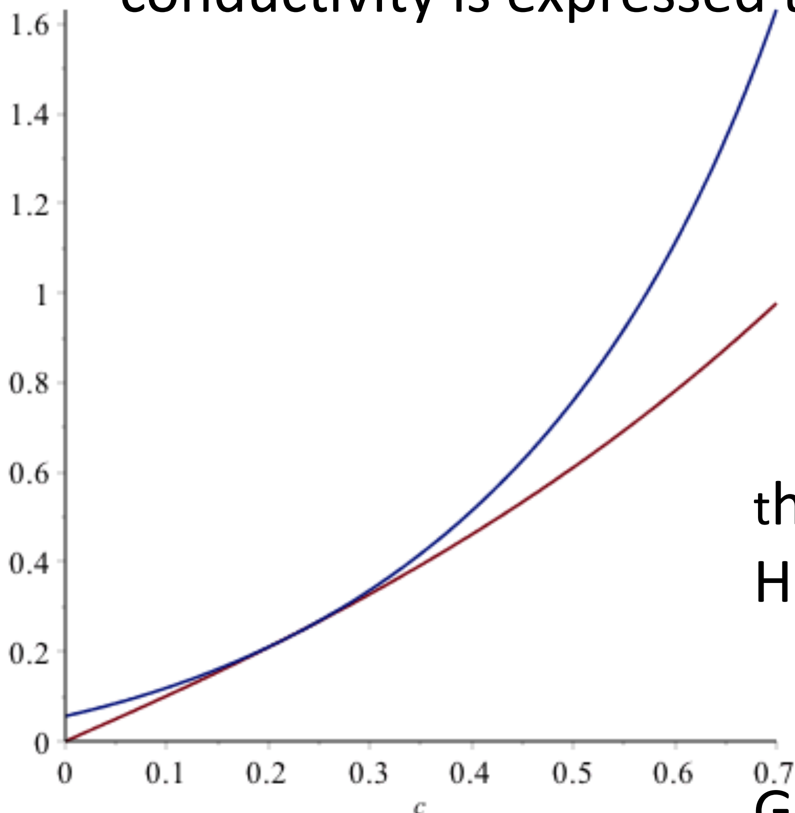
$$m_2 = \gamma^3,$$

$$m_1 = 3\gamma(1-\gamma)c\,\frac{ck_1(1-\gamma)+\gamma k_2(2-c)}{(2-c)k_2},$$

$$k_0 = \gamma c k_1 \frac{2ck_1(1-\gamma)+\gamma k_2(2-c)}{(2-c)(c\gamma k_1+(1-\gamma)k_2)}$$

# Optimality

With fixed *m2*, the structures varies with *m1*. The effective conductivity is expressed through *m1*. When




$$c = \frac{2k_2}{2k_1 + k_2} \text{ or}$$

$$m_1 = m_1^0 = \frac{2k_2}{2k_1 + k_2}\left(m_2^{1/3} - m_2\right)$$

the conductivity *k0* realizes Hashin–Shtrikman bound.

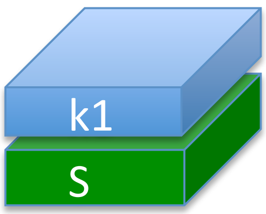
Graphs of the H-S bound and the effective Conductivity k0 of the bulk structures

# Larger *m1*: “Enveloping” theorem

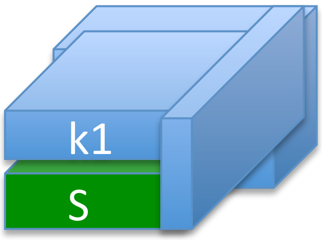
**Th:** If a structure S is translation–optimal, than a laminate form S and k1 is also translation–optimal.

$$\frac{1}{k_0+(d-1)k_1} \geq \sum_{i=1}^{N} \frac{m_i}{k_i+(d-1)k_1} \quad \text{- H-S bound}$$

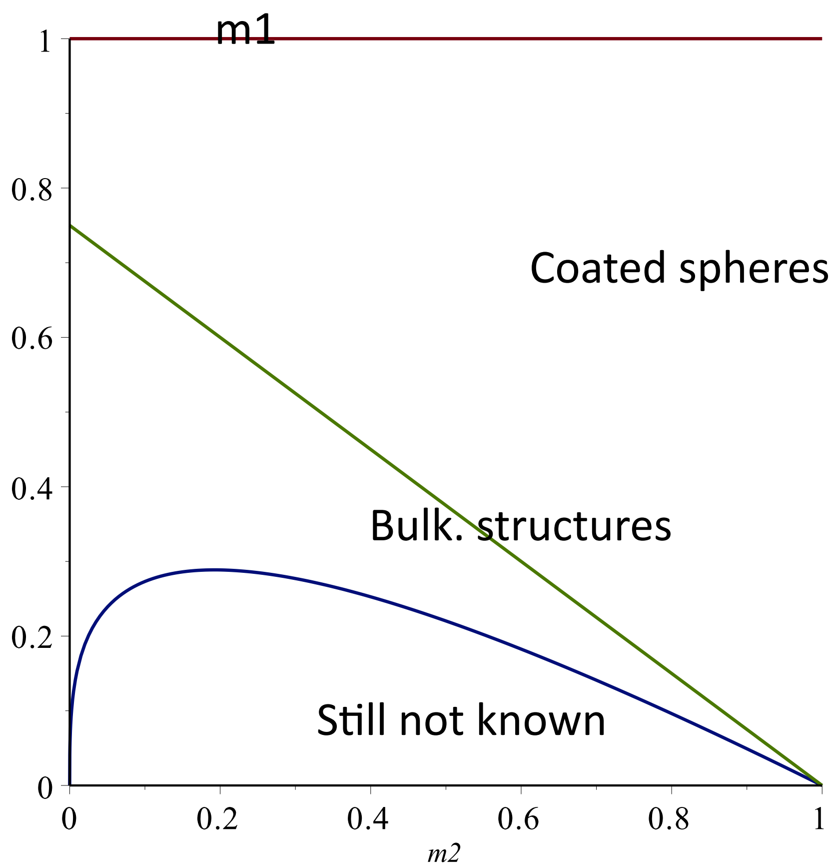
$$\mathrm{Tr}\left(K_0 - k_1 I\right)^{-1} \geq \sum_{i=2}^{N} \frac{m_i d}{k_i - k_1} + \frac{m_1}{k_1}, \; K_0 \leq \langle \mathrm{k} \rangle \mathrm{I} \text{ - } \quad \text{Translation bound}$$




Therefore, it is sufficient to look for a *critical optimal* structure with minimal amount of k1.



# Range of optimality



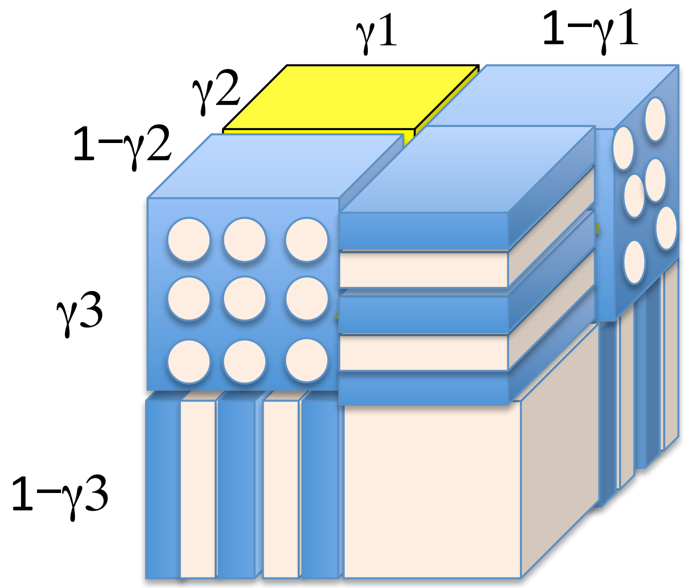

Additional relation on fields (dual variables) in the limiting structures

$$\det(e_1 - e_3) = 0 \ \forall x \in \Omega_1$$

Conjectured inequality:

$$\det(e_1 - e_3) \geq 0 \ \forall x \in \Omega_1$$

# Anisotropic optimal structures (similar to 2d consideration by Ch, Dzierzanowski, 2014)




- The consideration is similar but more messy.
- We show that this anisotropic structure satisfies translation bound at the same value of m1 for all values of gamma_i, if

$$m_2 = \gamma_1\gamma_2\gamma_3,$$

$$m_1 = g(\gamma_1,\gamma_2,\gamma_3)$$

- Optimality of degenerative anisotropic structures is conjuncted, but not proven

# P2. Dual problem of minimal resistance. Controllable Differential scheme

- **Problem:** Given three materials with resistances k1, k2, k3

$$k_1 < k_2 < k_3 = \infty$$

find a structure of minimal conductance (max resistance)

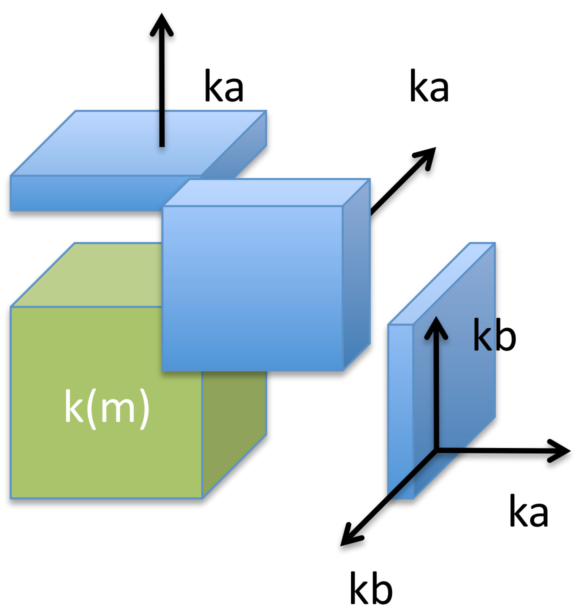
$$v_0^T k_0 v_0 + \gamma_0 = \inf \int_\Omega F(v)dx, \quad \nabla\cdot v = 0,$$

$$F = \min\left(v^T k_1 v + \gamma_1, v^T k_2 v + \gamma_2, v^T k_3 v\right)$$

*The bulk block structures are not optimal.*

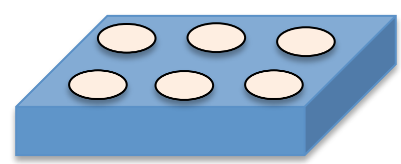
# Controllable differential scheme: Differential equation of structural evolution

- Three infinitesimal layers of volume *dm* of transversally isotropic composite is added in three orthogonal directions to an isotropic material “seed” with conductivity *k(m)*:

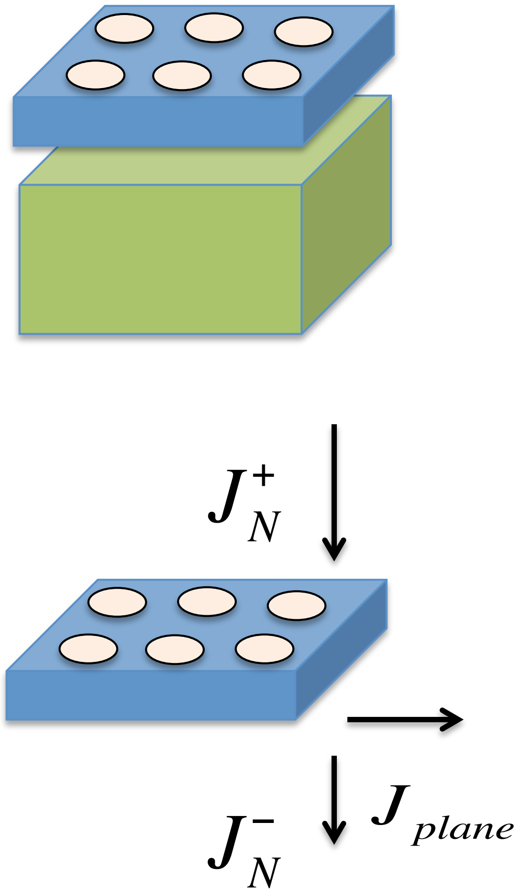



$$m\frac{dk}{dm}=\frac{(k_a-k)k}{3k_a}+\frac{2(k_b-k)}{3}$$

The added layers are differently oriented 2d-optimal composite



# Added composite of variable conductivity




- Assume that the added layer is a 2d-optimal composite with perfectly conducting cylindrical inclusions

$$k_a = \infty, \quad k_b = k_1\left(\tfrac{2}{c} - 1\right).$$

- The potentials at the upper and lower surface are equal, the current flows only in the plane.

- Notice, that the element can "turn" the current and transmit the potential

$$\left|J_N^+ - J_N^-\right| = \left|J_{plane}\right|, \quad u^+ = u^-$$

c=c(m) – the volume fraction of k1 in the composition –
Is a control (a variable parameter, that is adjusted to minimize the conductivity of the whole structure

# Optimization problem

Choose *c=c(m)* to minimize *k(1)* if

1. $m\dfrac{dk}{dm} = \dfrac{2k_b - k}{3}, \quad k_b = k_1\left(\dfrac{2}{c} - 1\right)$ - eq of struct. evolution

2. $k(m_2) = k_2$ - nucleous from $k_2$

3. $\int_{m_2}^{1} \mathrm{c}\,dm = m_1, \quad c(m) \in (0,1].$

Technique:

- Hamiltonian

$$H = \lambda(m)\frac{2k_1\left(\frac{2}{c} - 1\right) - k}{3m} + \beta c$$

Optimality conditions

$$\frac{d\lambda}{dm} = -\frac{\partial H}{\partial k}, \lambda(1) = 1,$$

$$\frac{\partial H}{\partial c} = 0 \Rightarrow c = c_{opt}(\beta) \Rightarrow \int_{m_2}^{1} \mathrm{c}_{opt}\,dm = m_1 \Rightarrow \beta = \beta_0.$$

# Optimization problem

Choose $c=c(m)$ to minimize $k(1)$

Solution:

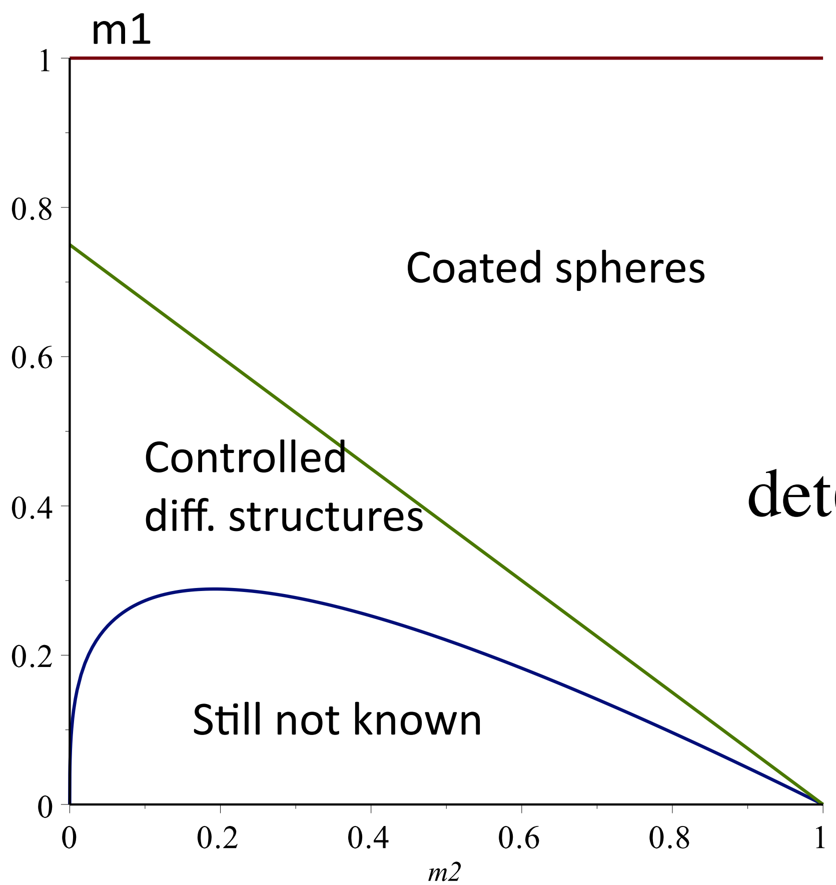
$$c_{opt}(m) = \frac{2m_1}{1 - m_2^{2/3}} m^{-1/3}$$

$$k_{opt} = m_2^{1/3} k_1 - 2k_1 \left(1 - m_2^{1/3}\right) = \frac{2k_1}{m_1} (m_2^{2/3} - 1)^2$$

$k_{opt}$ satisfies H-S bound, when

$$m_1 = m_1^{cr} = m_2^{1/3} \left(1 - m_2^{1/3}\right) \frac{3k_1}{k_2 + 2k_1}$$

# Range of optimality



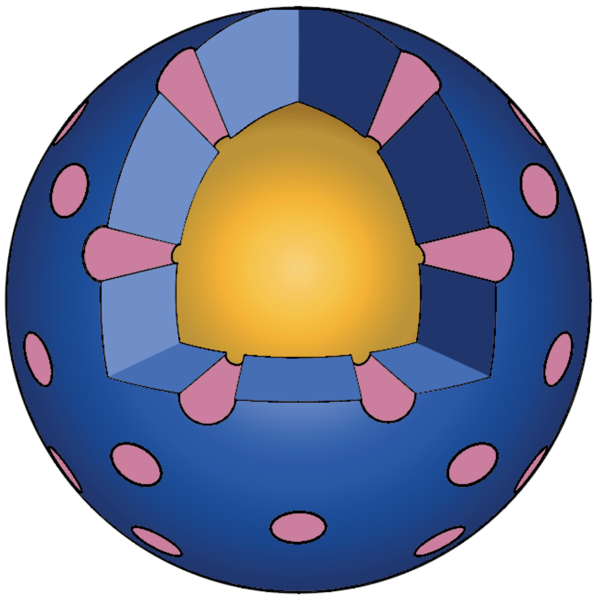

Additional relation on currents (dual variables) in the limiting structures

$$\det(j_1 - j_3) = 0 \text{ everywhere in } \Omega_1$$

Conjectured inequality:

$$\det(j_1 - j_3) \geq 0 \text{ everywhere in } \Omega_1$$

# Spherical layer and fields.



At the perforated layer:

$$u = \text{const}$$

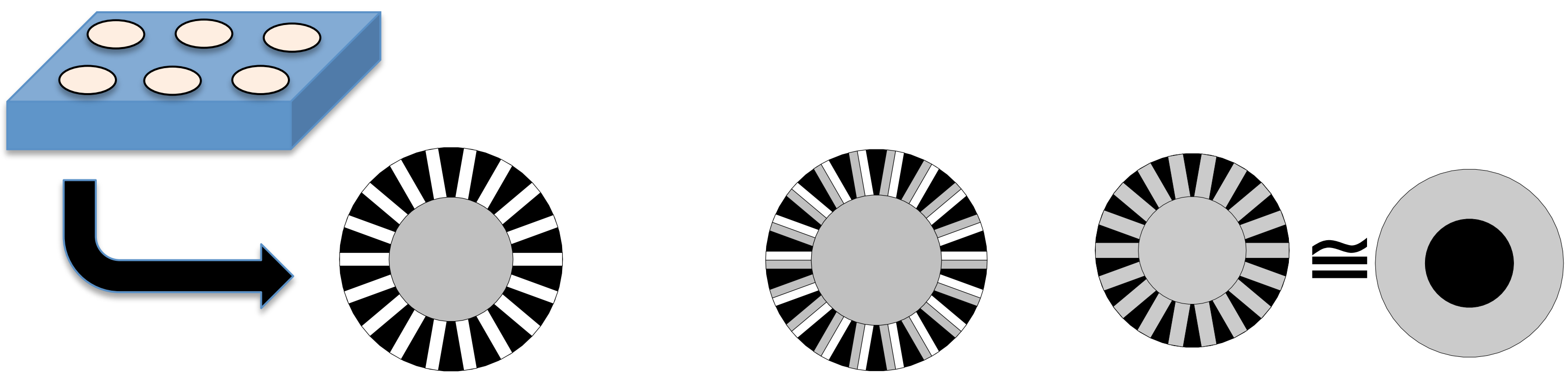
$$k_1 \frac{\partial u}{\partial r}\bigg|_{r=r1} - k_2 \frac{\partial u}{\partial r}\bigg|_{r=r2} = G(u)$$

- Using spherical coordinates and the hint from diff. scheme, we also can separate variables and compute properties of coated anisotropic spheres geometry, using effective field theory. The result is the same.

- Larger m1- add new outer layer from m1.

# Next evolution: Conjuncted structures

- In the limiting structure, the current in k1 is two–dimensional, the normal current is zero.
- Therefore, the next evolution of optimal structures, when m1 decreases, will likely follow the evolution of 2d optimal structures:



k2 enters the enveloping composite, moving from the central sphere. k1 moves into radial layers

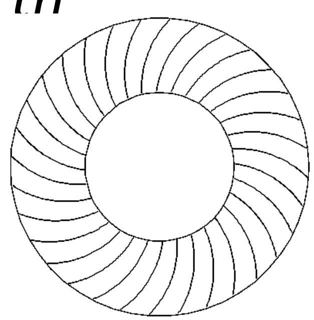
k2 also moves into radial layers, replaces k1

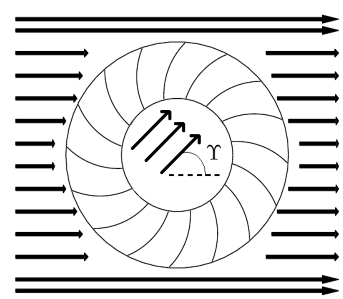
One can check, that the limiting k2-k3 structure is optimal and equivalent to coated spheres

# Metamaterials - structures with exotic properties

- **Bi-products of optimal structures study**:
- Using differential scheme and separation of variables in curved coordinates, one can create various geometries with unusual overall properties, inner degrees of freedom, abnormal and nonlinear etc.
- **Example:** A homogeneous external field applied to the spiral assemblies (Ch, Pruss, 13) with inclusions is transformed into a rotated homogeneous field inside inclusions.

*Sun rises from the North*





# Optimal design problem

(in collaboration with G. Dzierzanowski and N. Briggs)

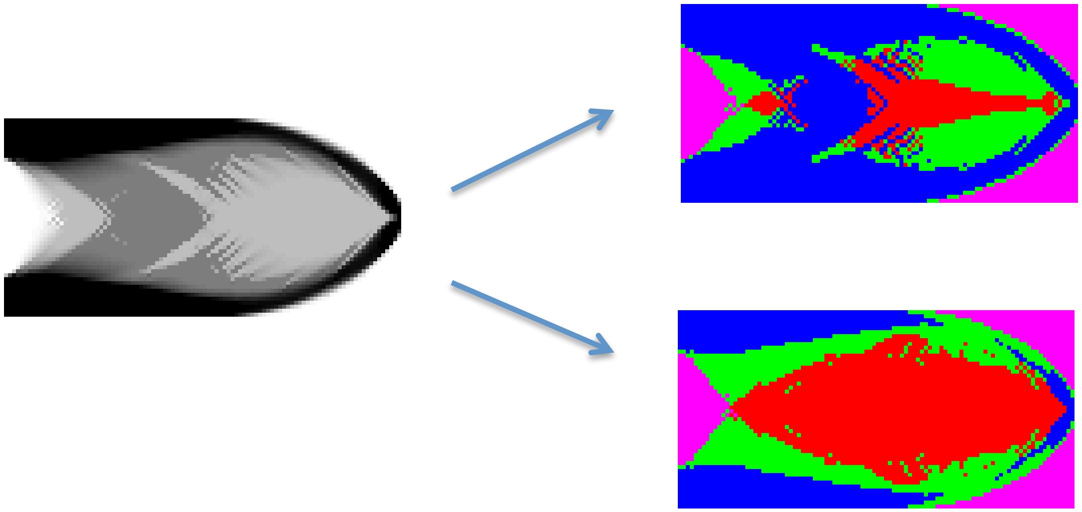
- **Optimal design problem:** determine optimal structures and distribute them in the design domain, keeping the total cost.
- **Three material design:** Given two materials – the strong expensive one, the cheap weak one, and the void, find their layout in a fixed loaded from the boundary domain, that minimizes the stress energy, i.e. maximizes the stiffness.

$$\inf_{\tau}\left(\int F(\boldsymbol{\tau},g)dx+\oint f(u^T\boldsymbol{\tau}n),\right) \quad \boldsymbol{\tau}:\ \boldsymbol{\tau}=\boldsymbol{\tau}^T,\nabla\cdot\boldsymbol{\tau}=0,$$

$$F=\min(W_i(\boldsymbol{\tau})+g_i), \quad W_i=k_i Tr(\boldsymbol{\tau}^2)/2$$

$$k_1<k_2<k_3=\infty, \quad g_1=1, \quad g_3=0, \quad g_2=\gamma,$$

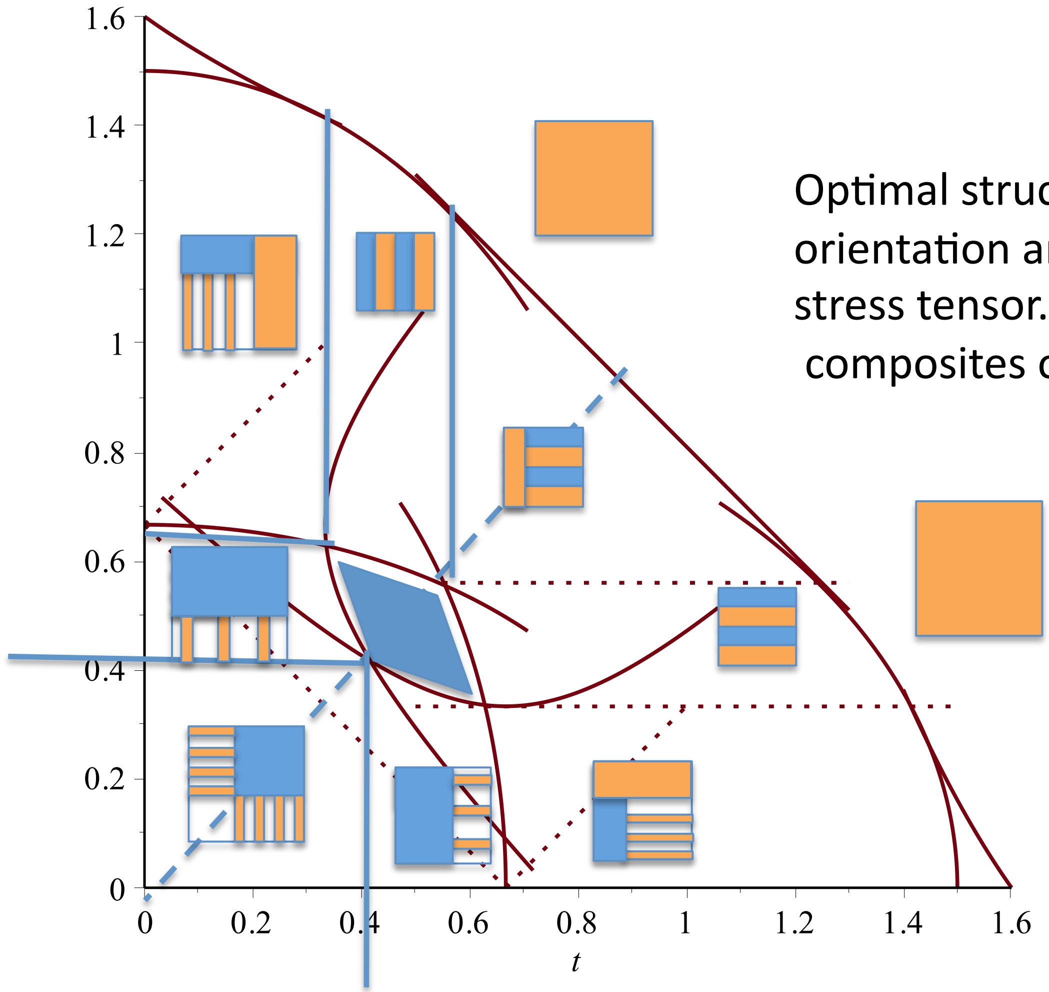
# Multimaterial design



Multimaterial designs are to two-material ones as color TV is to black–and–white TV.

| | | |
|---|---|---|
| Multimaterial design | | Color TV |
| ---------------------------- | = | ------------------------- |
| Two-material design | | Black-and-white TV |

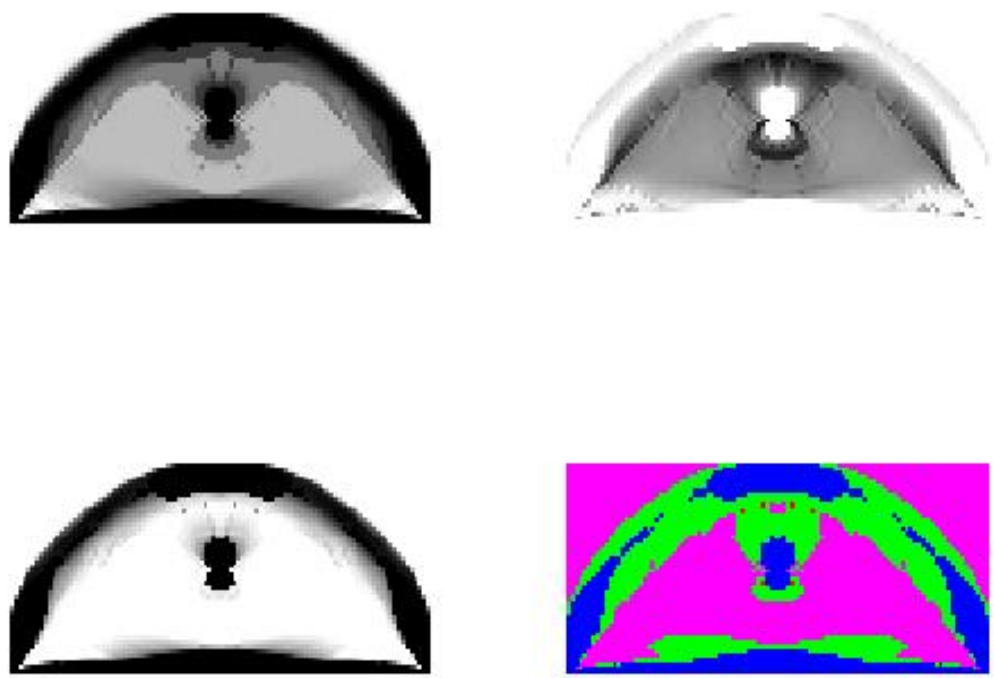
# Optimal structures and Quasiconvex envelope



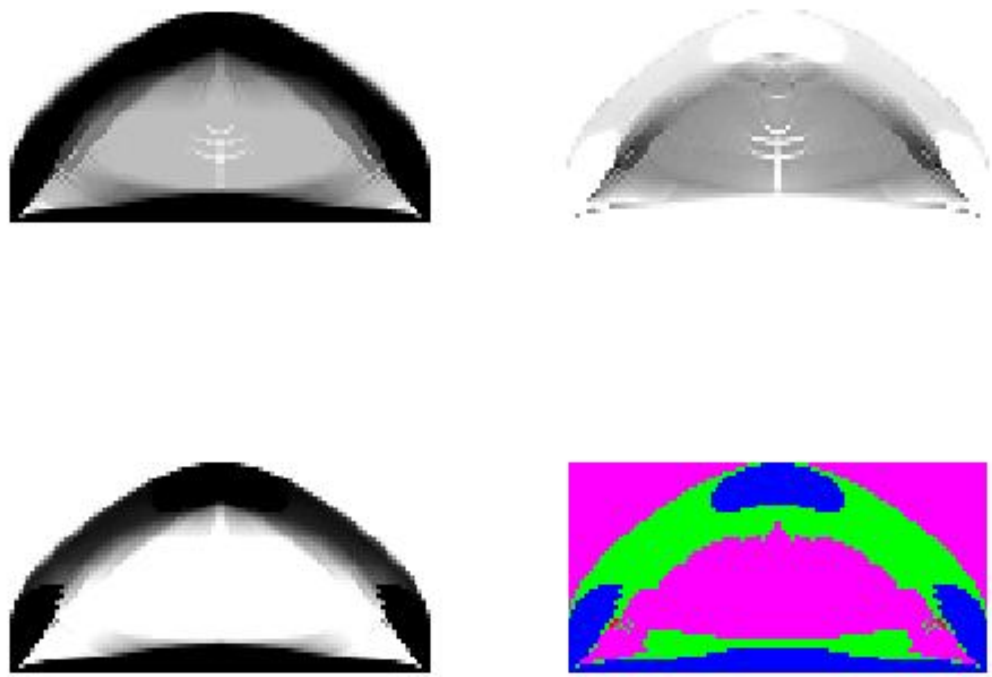

Optimal structures depend on intensity orientation and degree of anisotropy of stress tensor. The degenerate into two material composites or pure materials.

Quasiconvex envelope is a multifaced surface; we obtain analytic expressions for all the faces.

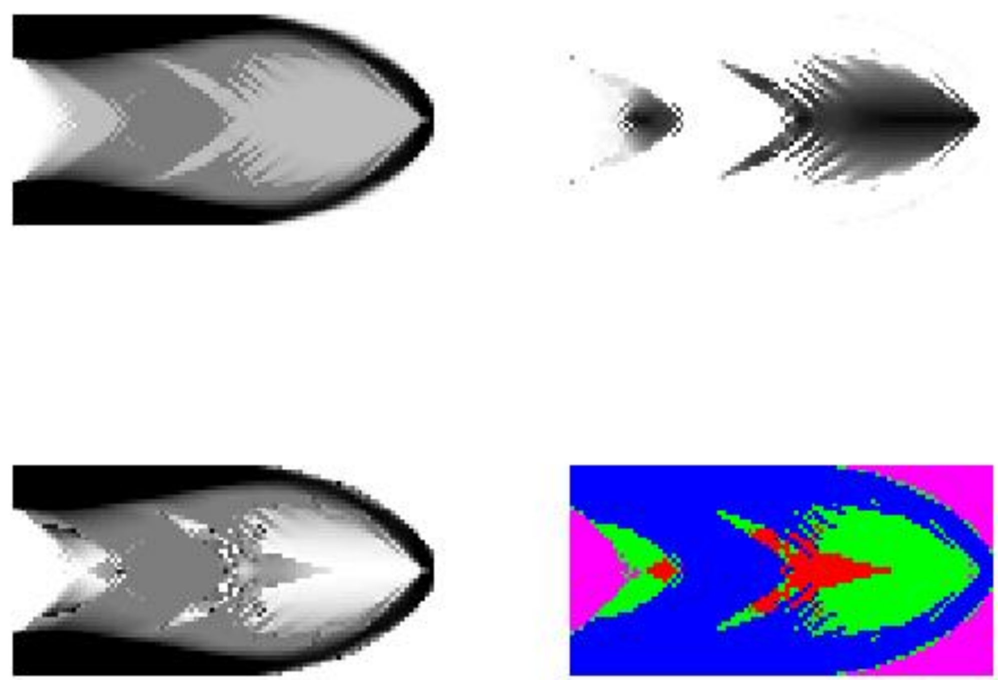
# Example of three-material design
# Bridge #1



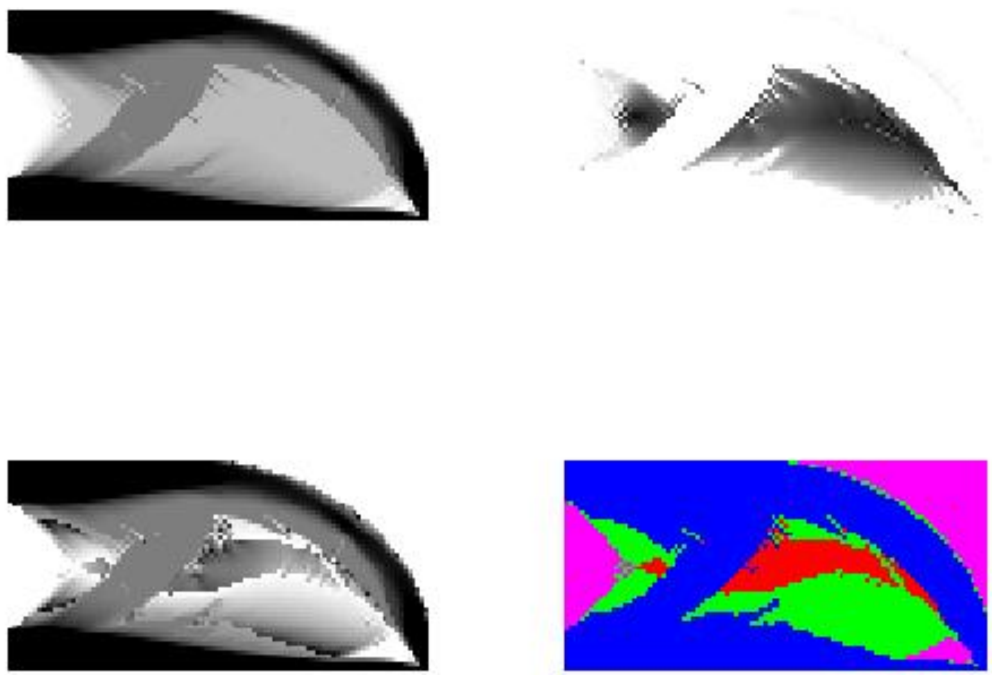
# Bridge #2, smaller total weight



# Cantilever, loaded at the middle



# Cantilever, loaded at the the edge



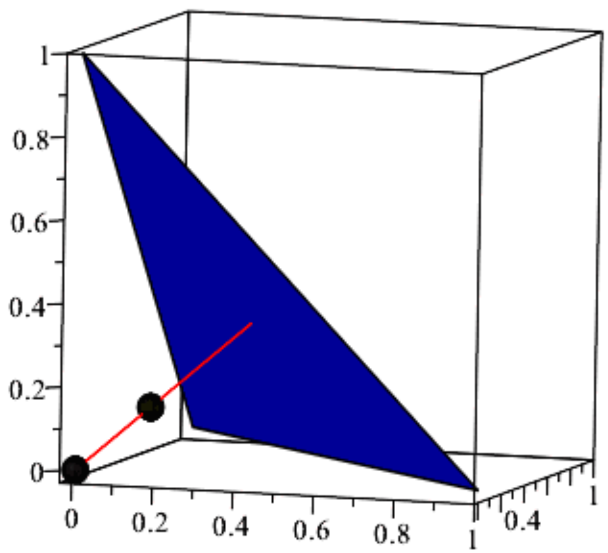
# Conclusion (List of unsolved problems)

- The technique for finding and proving inequalities on optimal fields needs to be developed.
- More examples of multimaterial optimal structures are needed.
- Techniques for several materials optimal design is to be developed and applied to elastic and electromagnetic materials.

- Quasiconvex envelope can be analytically described, 3d optimal structures are doable and cool.

Thank you!

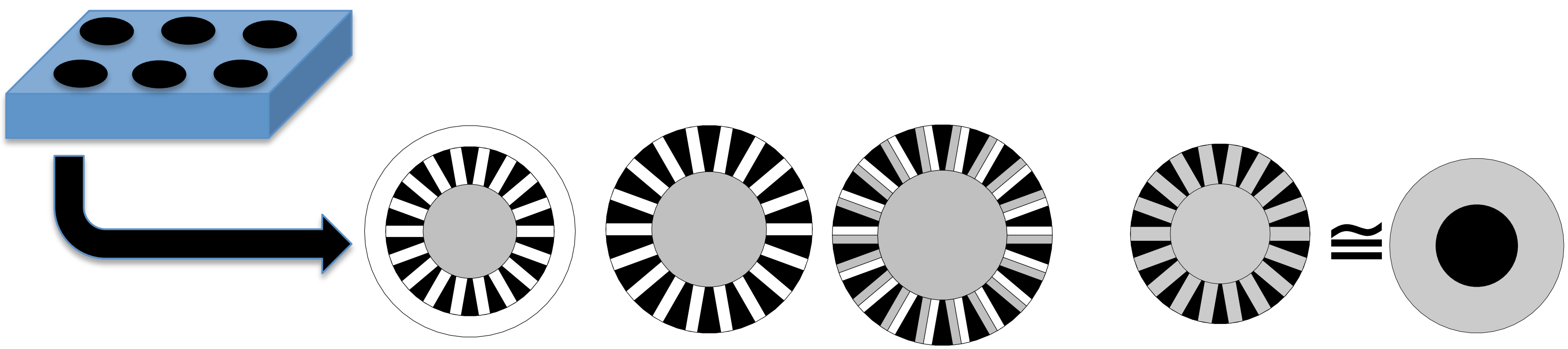
# Conjectured Range of the fields in an optimal 3d 3-material structure



***Thank you!***

# Next evolution: Conjuncted structures

- In the limiting structure, the current in k1 is two–dimensional, the normal current is zero.
- Therefore, the next evolution of optimal structures, when m1 decreases, likely will follow the evolution of 2d optimal structures:



k2 enters the enveloping composite, moving from the central part

k1 moves into radial layers

k2 also moves into radial layers, replaces k1

One can check, that the resulting k2-k3 structure is optimal and equivalent to coated spheres

# Bulk blocks assemblage - 2

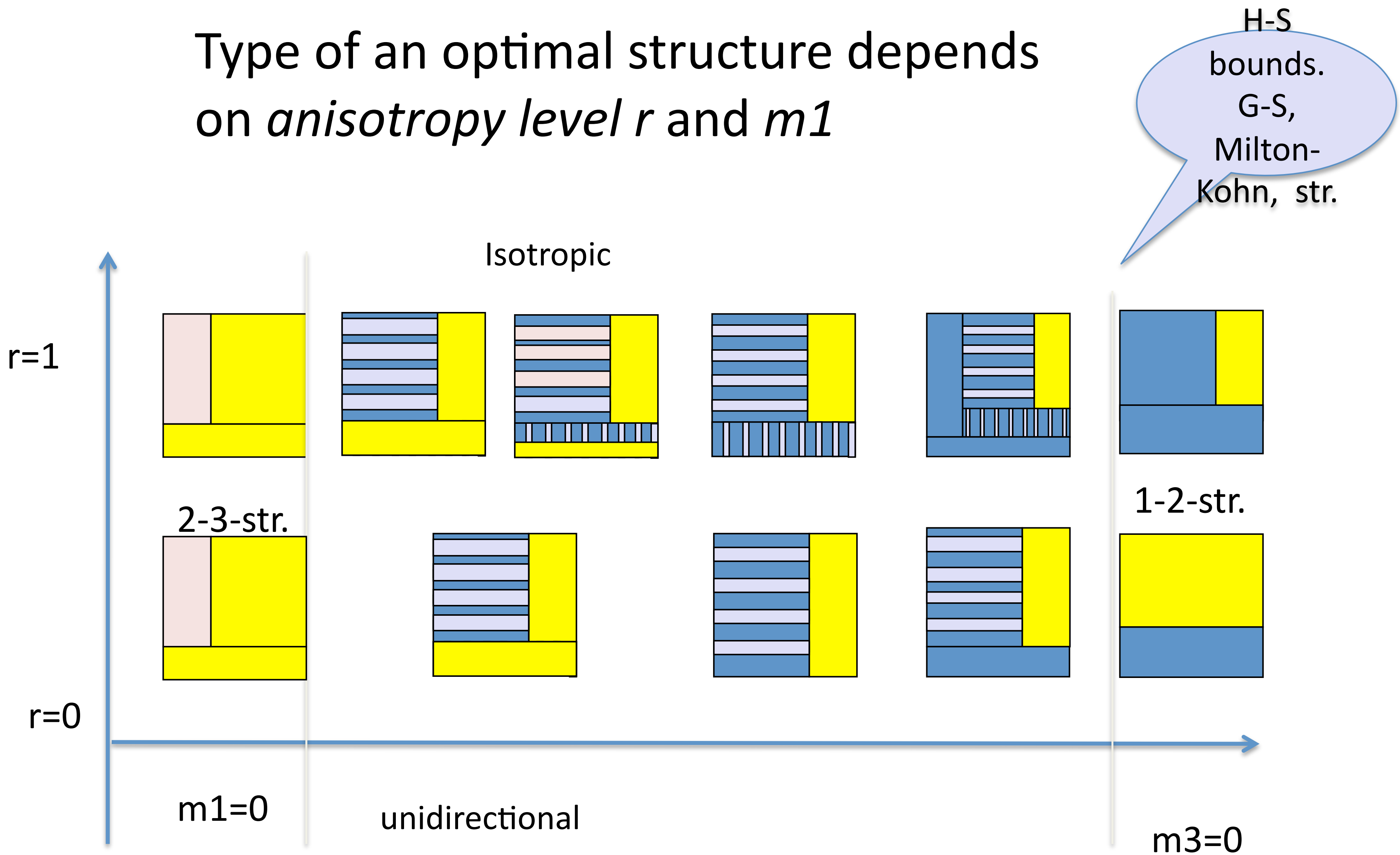
$$K2 = \begin{pmatrix} k_2 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_2 \end{pmatrix},$$

$$Kc = \begin{pmatrix} kca & 0 & 0 \\ 0 & kcb & 0 \\ 0 & 0 & kcb \end{pmatrix}, \quad kca = \frac{k_1}{c_c}, \quad kcb = k_1 \frac{2 - c_c}{c_c},$$

$$K3 = 0,$$

$$Kl = \begin{pmatrix} 0 & 0 & 0 \\ 0 & kl & 0 \\ 0 & 0 & kl \end{pmatrix}, \quad kl = k_1 c_l,$$

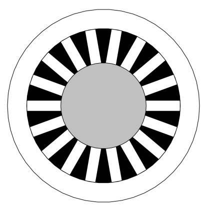
# Type of an optimal structure depends on *anisotropy level r* and *m1*

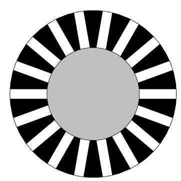


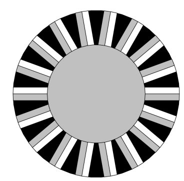

The structures realize sufficient optimality conditions: the variety is due to active/inactive point wise inequalities on the fields in opt. composite.

# Smaller m1: **conjecture.**

- Notice, that k1 is located in 2d cylindrical or 1d laminate substructures. When m1 decreases, k1 can be replaced by k2, as in 2d structures.







- One can check, that the limiting k2-k3 bulk structure realizes HS bounds for two materials $m_1 \to 0$ (is equivalent to the coated spheres)