Final Exam
M 5500, Spring 2019

Solve any six problems

Return at May 1, before 5 pm

1. Find Euler equation and natural boundary condition, check Legendre condition for the constrained problem

$$I(u) = \min_{u(x)} \int_0^1 \left((u')^2 - 3au\, u' + 3\sin(x) u^2 \right) dx \quad \text{if} \quad \int_0^1 (u)\, dx = 1$$

where a is a parameter.

2. Find the the curve connecting the origin $(0,0)$ with the line $y + x = 1$ along which the particle falls in the shortest time under the influence of gravity (brachistochrone).

3. Consider the problem

$$I = \min_{u(x)} \int_0^1 [a\,(u')^2 - c\,u(u-1)]dx, \quad u(0) = 0$$

where $a > 0$ and $c > 0$ are real parameters

a. Write Hamiltonian and dual Lagrangian.
b. Find an invariant.
c. Derive Euler equations and boundary conditions for the primary and the dual problems, derive system of first-order equation and boundary conditions.

4. A process is described by a differential equation

$$-au'' + cu^3 + 1 = 0, \quad \text{in } (0,1) \quad u(0) = 0, \; u(1) + u'(1) = 3$$

Write a variational problem (Lagrangian and the boundary term) for which this equation is the Euler equation.

5. Show that the Euler equation of the problem

$$\inf_{u(x)} \int_0^3 \left((u-x)^2 + \min\left\{(u')^2,\ (u'-2)^2+1\right\}\right) dx, \quad u(0)=0$$

does not satisfy the Weierstass test. Find the relaxed Lagrangian and the solution for the relaxed problem. Describe a minimizing sequence.

6. Find Euler-Lagrange equation and natural boundary conditions for the constained problem

$$I(n) = \min_{u(x)} \left[\int_\Omega \left((\nabla u)^2 - q\,u\right) dx + \oint_\Gamma \left((c\,u + d\,u^2\right) ds\right], \quad \int_\Omega u^2\, dx = 1$$

where $x = (x_1, x_2)$, Ω is a bounded domain in R_2 with the smooth boundary Γ, s is the coordinate along Γ, $u(x)$ is a scalar minimizer, $q,\ c,\ d$ are real parameters.

7. Describe the boundary of the cluster of two attached plane domains of equal area A if the total length of the boundaries of the cluster is minimal.