1 Introduction to Lagrangian mechanics

Leibnitz and Mautoperie suggested that any motion of a system of particles always minimizes a functional of action; later Lagrange came up with the exact definition of that action: the functional that has the Newtonian laws of motion as its Euler equation or stationarity condition. The question of whether the action reaches the actual minimum is complicated: Generally, it does not. We show below that the actual motion of particles reach either a local minimum or a saddle point of action. The variational formulation permits a regular derivation of a motion with Newtonian forces as an Euler equation of the action. The variational principles remain the abstract and economical way to describe Nature, but one should be careful in proclaiming the ultimate goal of the Universe.

1.1 Stationary Action Principle

Lagrange observed that the second Newton's law for the motion of a particle,

$$m\ddot{x} = f(x)$$

can be viewed as the Euler equation to the variational problem

$$\min_{x(t)} \int_{t_0}^{t_f} \left(T(\dot{x}) - V(x)\right) dx$$

that is:

$$m\ddot{x} + \frac{dV}{dx} = 0.$$

Here, V is the negative of antiderivative (potential) of the force f, and T is the kinatic energy of the particle

$$V = -\int f(x)dx, \quad T = \frac{1}{2}m(\dot{x})^2$$

The minimizing quantity – the difference between kinetic and potential energy – is called *action*; The Newton equation for a particle is the Euler equation for the integral of the action.

Example 1.1 (Pendulum) the kinetic energy K of a mass m is $K = \frac{1}{2}m\dot{x}^2$, the potential energy V of a spring is $V = \frac{1}{2}Cx^2$, where $x(t)$ is the deflection of the mass from the equilibrium position, C is is the stiffness of the spring. The action is

$$\int_{t_0}^{t_f} \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}Cx^2\right) dx$$

The Euler equation $m\ddot{x} + Cx = 0$ is the equation of pendulum.

In the stated form, the principle applies to any system of free interacting particles; one only need to specify the forms of kinetic and potential energies to obtain the Newtonian motion.

Example 1.2 (Central forces) The problem of celestial mechanics deals with system bounded by gravitational forces f_{ij} acting between any pair of masses m_i and m_j and equal to

$$f_{ij} = \gamma \frac{m_i m_j}{|r_i - r_j|^3}(r_i - r_j)$$

where vectors r_i define coordinates of the masses m_i as follows $r_i = (x_i, y_i, z_j)$. The corresponding potential V for the n-masses system is

$$V = -\frac{1}{2}\sum_{i,j}^{N} \gamma \frac{m_i m_j}{|r_i - r_j|}$$

where γ is Newtonian gravitational constant. The kinetic energy T is the sum of kinetic energies of the particles

$$T = \frac{1}{2}\sum_{i}^{N} m_i \dot{r}_i^2$$

The motion corresponds to the stationary value to the Lagrangian $L = T - V$, or the system of N vectorial Euler equations

$$m_i \ddot{r}_i - \sum_{j}^{N} \gamma \frac{m_i m_j}{|r_i - r_j|^3}(r_i - r_j) = 0$$

for N vector-function $r_i(t)$.

Since the Lagrangian is independent of time t, the first integral (**??**) exist

$$T + V = \text{ constant}$$

which corresponds to the conservation of the whole energy of the system.

Later in Section **??**, we will find other first integrals of this system and comment about properties of its solution.

Example 1.3 (Spring-mass system) Consider the sequence of masses located on an axis with coordinates $m_1, \ldots, m_n$; the neighboring masses are joined by the springs. Each spring generate force f_i proportional to $x_i - x_{i+1}$ x_i is the deflection of ith mass from the equilibrium position.

The equations of motion of this system is as follows. The kinetic energy T of the system is equal to the sum of kinetic energies of the masses,

$$T = \frac{1}{2}m(\dot{x}_1 + \ldots + \dot{x}_n).$$

The potential energy V is the sum of energies of all springs, or

$$V = \frac{1}{2}C_1(x_2 - x_1)^2 + \ldots + \frac{1}{2}C_{n-1}(x_n - x_{n-1})^2$$

The Lagrangian $L = T - V$ correspond to n differential equations

$$\begin{aligned}
m_1\ddot{x}_1 &+ C_1(x_1 - x_2) = 0 \\
m_2\ddot{x}_2 &+ C_2(x_2 - x_3) - C_1(x_1 - x_2) = 0 \\
\dots &\quad \dots \\
m_n\ddot{x}_n &- C_{n-1}(x_{n-1} - x_n) = 0
\end{aligned}$$

or in vector form

$$M\ddot{x} = P^T C P x$$

where $x = (m_1, \dots, x_n)$ is the vector of displacements, M is the $n \times n$ diagonal matrix of masses, V is the $(n-1) \times (n-1)$ diagonal matrix of stiffness,

$$M = \begin{pmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & m_n \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & 0 & \dots & 0 \\ 0 & C_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & C_{n-1} \end{pmatrix}$$

and P is the $n \times (n-1)$ matrix that corresponds to the operation of difference,

$$P = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

When the masses and the springs are identical, $m_1 = \dots = m_n = m$ and $C_1 = \dots = C_{n-1} = C$, the system simplifies to

$$\begin{aligned}
m_1\ddot{x}_1 &+ C(x_1 - x_2) = 0 \\
m_2\ddot{x}_2 &+ C(-x_1 + 2x_2 - x_3) = 0 \\
\dots &\quad \dots \\
m_n\ddot{x}_n &- C(x_{n-1} - x_n) = 0
\end{aligned}$$

or in vector form,

$$\ddot{x} + kP_2 x = 0$$

where $k = \frac{C}{m}$ is the positive parameter, and $P_2 = P^T P$ is the $n \times n$ matrix of second differences,

$$P_2 = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

This system of linear equations can be solved by the standard methods.

1.2 Generalized coordinates

The Lagrangian concept allows for obtaining equations of motion of a constrained system. In this case, the kinetic and potential energy must be defined as a function of *generalized coordinates* that describes degrees of freedom of motion consistent with the constraints. If a particle can move along a surface, one can define coordinates on this surface and allow the motion only along these coordinates. The kinetic and potential energies must be expressed via these coordinates.

The particles can move along the generalized coordinates q_i. The number of the generalized coordinates q_i corresponds to the degrees of freedom. The consistent with constraints position x becomes $x(q)$. The speed $\dot{x}$ becomes a linear form of $\dot{q}$

$$\dot{x} = \sum \left(\frac{\partial x}{\partial q_i} \dot{q}_i \right)$$

For example, a particle can move along the circumference of the radius R, the generalized coordinate will be an angle θ which determines the position $x_1 = R \cos\theta$, $x_2 = R \sin\theta$ at this circumference and its speed becomes

$$\dot{x}_1 = -R\dot{\theta} \sin\theta, \quad x\dot{x}_2 = R\dot{\theta} \cos\theta$$

This system has only one degree of freedom, because fixation of one parameter θ completely defines the position of a point at a circumference.

When the motion is written in terms of generalized coordinates, the constraints are automatically satisfied. The potential and kinetic energies can be expressed though these coordinates. The potential energy $V(x)$ is rewritten as $W(q) = V(x(q))$ and the kinetic energy $T(\dot{x}) = \sum_i m_i \dot{x}_i^2$ becomes a quadratic form of derivatives of generalized coordinates $\dot{q}_i = \frac{\partial x}{\partial q_i} \dot{q}_i$:

$$T(\dot{x}) = \frac{1}{2} \sum_i m_i \dot{x}_i^2 = \frac{1}{2} \dot{\boldsymbol{q}}^T R(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

where the symmetric nonnegative matrix R is equal to

$$R = \{R_{ij}\}, \quad R_{ij} = \left(\frac{\partial T}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial q_i} \right)^T \left(\frac{\partial T}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial q_j} \right)$$

Notice that $T_q(\dot{q})$ is a homogeneous quadratic function of $\dot{q}$, $T_q(k\dot{q}) = k^2 T_q(\dot{q})$ and therefore

$$\frac{\partial}{\partial \dot{\boldsymbol{q}}} T_q(\boldsymbol{q}, \dot{\boldsymbol{q}}) \cdot \dot{\boldsymbol{q}} = 2T_q(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{1}$$

The variational problem that correspond to minimal action with respect to generalized coordinates becomes

$$\min_{\boldsymbol{q}} \int_{t_0}^{t_1} (T_q - V_q) dt \tag{2}$$

Because potential energy V does not depend on $\dot{q}$, the Euler equations have the form

$$\frac{d}{dt}\frac{\partial T_q}{\partial \dot{\boldsymbol{q}}} - \frac{\partial}{\partial \boldsymbol{q}}(T_q - V_q) = 0 \tag{3}$$

which is similar to the form of unrestricted motion.

When the Lagrangian is independent of t the system is called *conservative*, In this case, the Euler equation assumes the first integral in the form (use (1))

$$\dot{\boldsymbol{q}}\frac{\partial T_q}{\partial \dot{\boldsymbol{q}}} - (T_q - V_q) = T_q + V_q = \text{constant}(t) \tag{4}$$

The quantity $\Pi = T_q + V_q$ is called the whole energy of a mechanical system; it is preserved along the trajectory of a conservative system.

The generalized coordinates help to formulate differential equations of motion of the constrained system. Consider several examples.

Example 1.4 (Isochrone) Consider a motion of a heavy mass along the cycloid:

$$x = \theta - \cos\theta, \quad y = \sin\theta$$

To derive the equation of motion, we write down the kinetic T and potential V energy of the mass m, using $q = \theta$ as a generalized coordinate. We have

$$T = \frac{1}{2}m\dot{x}^2 + \dot{y}^2 = m(1 + \sin\theta)\dot{\theta}^2$$

and $V = -my = -m\sin\theta$.

The Lagrangian

$$L = T - V = m(1 + \sin\theta)\dot{\theta}^2 + m\sin\theta$$

corresponds to the Euler equation

$$S_L(\theta) = \frac{d}{dt}\left((1 + \sin\theta)\frac{d\theta}{dt}\right) - \cos\theta = 0.$$

which solution (check with Maple!) is

$$\theta(t) = \arccos(C_1 \sin t + C_2 \cos t)$$

where C_1 and C_2 are constant of integration. One can check that $\theta(t)$ is 2π-periodic for all values of C_1 and C_2. This explains the name "isochrone" given to the cycloid before it was found that this curve is also the brachistochrone (see Section **??**)

Example 1.5 (Winding around a circle) Describe the motion of a mass m tied to a cylinder of radius R by a rope that winds around it when the mass evolves around the cylinder. Assume that the thickness of the rope is negligible small compared with the radius R, and neglect the gravity.

It is convenient to use the polar coordinate system with the center at the center of the cylinder. Let us compose the Lagrangian. The potential energy is zero, and the kinetic energy is

$$
\begin{aligned}
L = T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \\
= \frac{1}{2}m\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)^2 + \frac{1}{2}m\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)^2 \\
= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)
\end{aligned}
$$

The coordinates $r(t)$ and $\theta(t)$ are algebraically connected by Pythagorean relation $R^2 + l(t)^2 = r(t)^2$ at each time instance t. Here $l(t)$ is the part of the rope that is not winded yet; it is expressed through the angle $\theta(t)$ and the initial length l_0 of the rope, $l(t) = l_0 - R\theta(t)$. We obtain

$$
(l_0 - R\theta(t))^2 = r(t)^2 - R^2 \quad \forall t \in [0, t_{\text{final}}],
$$

and observe that the time of winding t_{final} is finite. The trajectory $r(\theta)$ is a spiral.

The obtained relation allows for linking of $\dot{r}$ and $\dot{\theta}$. We differentiate it and obtain

$$
r\dot{r} = -R(l_0 - R\theta(t))\dot{\theta} = -R(\sqrt{r^2 - R^2}\dot{\theta}
$$

or

$$
\dot{\theta} = -\frac{\dot{l}}{R} = -\frac{r\dot{r}}{R\sqrt{r^2 - R^2}}
$$

The Lagrangian becomes

$$
L(r, \dot{r}) = \frac{1}{2}m\dot{r}^2\left(1 + \frac{r^4}{R^2(r^2 - R^2)}\right)
$$

Its first integral

$$
\frac{1}{2}m\dot{r}^2\left(1 + \frac{r^4}{R^2(r^2 - R^2)}\right) = C
$$

shows the dependence of the speed $\dot{r}$ on the coordinate r. It can be integrated in a quadratures, leading to the solution

$$
t(r) = C_1 \int_{r_0}^{r} \sqrt{\frac{r^2 - R^2}{r^4 + R^2 r^2 - R^4}}\, dr
$$

The two constants r_0 and C_1 are determined from the initial conditions.

The first integral allows us to visualize the trajectory by plotting $\dot{r}$ versus r. Such graph is called the phase portrait of the trajectory.

Example 1.6 (Move through a funnel) Consider the motion of a heavy particle through a vertical funnel. The axisymmetric funnel is described by the equation

$z = \phi(r)$ in cylindrical coordinate system. The potential energy of the particle is proportional to z, $V = -mgz = -mg\phi(r)$ The kinetic energy is

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right)$$

or, accounting that the point moves along the funnel,

$$T = \frac{1}{2} m \left((1 + \phi'^2) \dot{r}^2 + r^2 \dot{\theta}^2 \right).$$

The Lagrangian

$$L = T - V = \frac{1}{2} m \left((1 + \phi'^2) \dot{r}^2 + r^2 \dot{\theta}^2 \right) + mg\phi(r)$$

is independent of the time t and the angle θ, therefore two first integrals exist:

$$\frac{\partial L}{\partial \dot{\theta}} = \mu \quad \Rightarrow \; \dot{\theta} = \frac{\mu}{r^2}$$

where μ is a constant, and

$$T + V = \frac{1}{2} m \left((1 + \phi'^2) \dot{r}^2 + r^2 \dot{\theta}^2 \right) - mg\phi(r) = \Pi$$

The last equation can be simplified by excluding $\dot{\theta}$ using the previous one,

$$\Pi = \frac{1}{2} m \left((1 + \phi'^2) \dot{r}^2 + \frac{\mu^2}{r^2} - g\phi(r) \right)$$

Here, the constants Π and μ can be defined from the initial conditions. They represent, respectively, the whole energy of the system and the angular momentum; these quantities are conserved along the trajectory. These integrals alone allow for integration of the system, without computing the Euler equations. Solving for $\dot{r}$, we find

$$\dot{r}^2 = 2 \frac{\left(\frac{2\Pi}{m} + g\phi(r) \right) r^2 - \mu^2}{1 + \phi'^2}$$

Consequently, we can find $r(t)$ and $\theta(t)$ (see Problem ??.

$$t = \int^r \sqrt{\frac{1 + \phi'^2}{2 \left(\frac{2\Pi}{m} + g\phi(r) \right) r^2 - \mu^2}} \, dr$$

A periodic trajectory corresponds to constant value $\dot{\theta}(t)$ and constant value of $r(t) = r_0$ which is defined by the initial energy, angular momentum, and the shape $\phi(r)$ of the funnel, and satisfies the equation

$$\frac{\mu^2}{r_0^2} - g\phi(r_0) = \frac{2\Pi}{m}$$

This equation does not necessary has a solution. Physically, a heavy particle can either tend to a stready revolutions around the funnel, or fall through it.

2 Weak and strong local minima

2.1 Norms in functional space

The answer to the question of whether or not two curves are close to each other depends on the definition of closeness. This question is studied in the theory of topological spaces. Unlike the distance between two points in finite-dimensional Euclidian space, the same two curves can be considered to be infinitesimally close or far parted depending on the meaning of "distance." The variational tests examine the stability of the stationary solutions to small perturbations; different tests differently define the smallness of perturbation.

In calculus of variations, there are three mostly used criteria to measure the closeness of two differentiable functions $f_1(x)$ and $f_2(x)$: The norm $\mathcal{N}_1$ of difference $\delta f(x) = f_1(x) - f_2(x)$ in the values of functions

$$\mathcal{N}_1(\delta f) = \max_{x\in(0,1)} |\delta f(x)|$$

the norm $\mathcal{N}_2$ of difference of their derivatives,

$$\mathcal{N}_2(\delta f) = \max_{x\in(0,1)} |\delta f'(x)|$$

and the length $\mathcal{N}_3$ of the interval on which these functions are different

$$\mathcal{N}_3(\delta f) = \Delta \quad \text{if} \;\; \delta f(x) = 0 \;\forall x \notin [x, x+\Delta]$$

None of the variational tests guarantees the global optimality of the tested trajectory, only local minimum; on the other hand, these tests are simple enough to be applied. The *local minimum* satisfies the inequality

$$I(u) \leq I(u+\delta u) \quad \forall \delta u : \; \mathcal{N}(\delta u(x)) < \varepsilon$$

where ε is infinitesimally small, and $\mathcal{N}$ is a norm. The definition of what is local minimum depends on the above definitions of the norm $\mathcal{N}$.

If the perturbation is small in the following sense

$$\mathcal{N}_{\text{Legendre}}(\delta u) = \mathcal{N}_1(\delta u) + \mathcal{N}_2(\delta u) + \mathcal{N}_3(\delta u) < \varepsilon$$

the Legendre text is satisfied. The test assumes that the compared functions and their derivatives are close everywhere, and they are identical outside of an infinitesimal interval.

The Weierstrass text assumes that the compared functions are close everywhere, and they are identical outside of an infinitesimal interval, but their derivatives are not close in the interval of variation:

$$\mathcal{N}_{\text{Weierstrass}}(\delta u) = \mathcal{N}_1(\delta u) + \mathcal{N}_3(\delta u) < \varepsilon.$$

If the objective functional satisfy the Weierstrass test, we say that the extremal $u(x)$ realizes a *strong local minimum.* The Weierstrass test is stronger than the Legendre test.

The Jacobi test (see below) assumes that

$$\mathcal{N}_{\text{Jacobi}}(\delta u) = \mathcal{N}_1(\delta u) + \mathcal{N}_2(\delta u) < \varepsilon$$

that is the compared functions and their derivatives are close everywhere, but the variation is not localized. The Jacobi test is stronger than the Legendre test. If Jacobi test is satisfied we say that the extremal $u(x)$ realizes a *weak local minimum* (not to be confused with the minimum of weakly convergent sequence and with the minimum for localized variations). Neither Weierstrass and Jacobi tests are stronger than the other: They test the stationary trajectory from different angles.

2.2 Sufficient condition for the weak local minimum

We assume that a trajectory $u(x)$ satisfies the stationary conditions and Legendre condition. Let us apply a nonlocal variation of an infinitesimal magnitude: $\delta u = O(\epsilon) \ll 1$ and $\delta u' = O(\epsilon) \ll 1$; we compute the expansion of the increment δI of the functional keeping terms up to $O(\epsilon^2)$. We also recall that the linear of ϵ terms are zero because the Euler equation for $u(x)$ holds. We have

$$\delta I = \int_0^r \left[\frac{\partial^2 F}{\partial u^2}(\delta u)^2 + 2\frac{\partial^2 F}{\partial u \partial u'}(\delta u)(\delta u') + \frac{\partial^2 F}{\partial (u')^2}(\delta u')^2\right] dx + o(\epsilon^2) \tag{5}$$

No variation of this kind can improve the stationary solution if the quadratic form

$$Q(u,u') = \begin{pmatrix} \frac{\partial^2 F}{\partial u^2} & \frac{\partial^2 F}{\partial u \partial u'} \\ \frac{\partial^2 F}{\partial u \partial u'} & \frac{\partial^2 F}{\partial (u')^2} \end{pmatrix}$$

is positively defined,

$$Q(u,u') > 0 \quad \text{on the stationary trajectory } u(x) \tag{6}$$

This condition is called the sufficient condition for the weak minimum because it shows that all sufficiently close and smooth curves cannot improve the cost of the problem compared with the stationary curve.

Notice that the term $\frac{\partial^2 F}{\partial u'^2}$ is non-negative because of the Legendre condition (**??**).

Example 2.1 Show that the sufficient condition is satisfied for the Lagrangians

$$F = \frac{1}{2}u^2 + \frac{1}{2}(u')^2 \text{ and } F_2 = \frac{1}{|u|}(u')^2$$

Next example shows that violation of the sufficient conditions can yield to nonexistance of the solution.

Example 2.2 (Nonexistence of the minimizer: Blow up) Consider the problem: Minimize the functional

$$I = \min_u \int_0^r \left(\frac{1}{2}(u')^2 - \frac{c^2}{2}u^2\right) dx \quad u(0) = 0; \quad u(r) = A$$

The first variation δI is zero,

$$\delta I = \int_0^r \left(u'' + c^2 u\right) \delta u dx = 0$$

if $u(x)$ satisfies the Euler equation

$$u'' + c^2 u = 0, \quad u(0) = 0, u(r) = A. \tag{7}$$

The Weierstrass test is satisfied, because the dependence on the derivative is convex.

The second variation is

$$\delta^2 I = \int_0^r \left(\frac{1}{2}(\delta u')^2 - \frac{c^2}{2}(\delta u)^2\right) dx$$

Since the ends of the trajectory are fixed, the variation δu satisfies homogeneous conditions $\delta u(0) = \delta u(r) = 0$. Let us choose the variation as follow:

$$\delta u = \begin{cases} \epsilon \sin\left(\frac{\pi x}{L}\right), & 0 \le x \le L \\ 0 & x > L \end{cases}$$

where the interval of variation $[0, L]$ is not greater that $[0, r]$, $L \le r$. Computing the second variation, we obtain

$$(\delta u)' = \begin{cases} \epsilon \frac{\pi}{L} \cos\left(\frac{\pi x}{L}\right), & 0 \le x \le L \\ 0 & x > L \end{cases}$$

and

$$\delta^2 I = \frac{1}{L}\epsilon^2 \left(\frac{\pi^2}{L^2} - c^2\right), \quad L \le r$$

We are specially interested in the dependence of optimal solution on r. The second variation is negative when r is large,

$$\delta^2 I \le \frac{1}{r}\epsilon^2 \left(\frac{\pi^2}{r^2} - c^2\right) < 0 \quad \text{if } r > c\pi$$

which shows that the a stationary solution is not a minimizer.

To clarify the phenomenon, let us compute the stationary solution from the Euler equation (7). We have

$$u(x) = \left(\frac{A}{\sin(cr)}\right) \sin(cx) \quad \text{and } I(u) = \frac{A^2}{\sin^2(cr)} \left(\frac{\pi^2}{r^2} - c^2\right)$$

When r increases approaching the value $c\pi$, the magnitude of the stationary solution indefinitely grows, and the cost indefinitely decreases:

$$\lim_{r \to \pi c - 0} I(u) = -\infty$$

The solution of the Euler equation that corresponds to finite $I(u)$ when $r > \pi c$ is not a minimizer.

3 Jacobi variation

The Jacobi condition examines the optimality of "long" trajectories. It complements the Weierstrass test that investigates the stability of a Lagrangian to strong localized variations. Jacobi condition tries to disprove optimality of a stationary trajectory by testing the dependence of Lagrangian on the minimizer itself not of its derivative. We assume that a trajectory $u(x)$ satisfies the stationary conditions and Weierstrass condition but does not satisfy the sufficient conditions for a weak minimum, $Q(u, u')$ is not positively defined.

To derive Jacobi condition, we apply again an infinitesimal nonlocal variation: $\delta u = O(\epsilon) \ll 1$ and $\delta u' = O(\epsilon) \ll 1$ and examine the expression (5) for the second variation. Notice that we denote the upper limit of integration in (5) by r; we are testing the stability of the trajectory depending on its length. When a nonlocal "shallow" variation is applied, the increment increases because of assumed positivity of $\frac{\partial^2 F}{\partial (u')^2}$ and decreases because of assumed nonpositivity of the matrix Q. Depending on the length r of the interval of integration and chosen form of the variation δu, one of these effects prevails. stronger, the extremal fails the test and is nonoptimal.

Let us choose the best shape δu of the variation. The expression (5) itself is a variational problem for δu which we rename as v; the Lagrangian is quadratic of v and v' and the coefficients are functions of x determined by the stationary trajectory $u(x)$:

$$\delta I = \int_0^r \left[A v^2 + 2B\, v\, v' + C (v')^2 \right] dx, \quad v(0) = v(r) = 0 \tag{8}$$

where

$$A = \frac{\partial^2 F}{\partial u^2}, \quad B = \frac{\partial^2 F}{\partial u \partial u'}, \quad C = \frac{\partial^2 F}{\partial (u')^2}$$

The problem (8) correspond to the Euler equation that is a solution to Storm-Liuville problem:

$$\frac{d}{dx}(C v' + B v) - A v = 0, \quad v(0) = v(r_{\text{conj}}) = 0 \quad \text{if } r < r_{\text{conj}} \tag{9}$$

with boundary conditions $v(0) = v(r) = 0$. The point r_{conj} is called a conjugate point to the end of the interval. The problem is homogeneous: If $v(x)$ is a solution and c is a real number, $cv(x)$ is also a solution.

Jacobi condition is satisfied if the interval does not contain conjugate points, that is there is no nontrivial solutions to (9) on any subinterval of $[0, r_{\text{conj}}]$, that is if there are no nontrivial solutions of (9) with boundary conditions $v(r) = v(r_{\text{conj}}) = 0$ where $0 \le r_{\text{conj}} \le r$.

If this condition is violated, than there exist many trajectories

$$u(x) \begin{cases} u_0 + v & \text{if } \ x \in [0, r_{\text{conj}}] \\ u_0 & \text{if } \ x \in [r_{\text{conj}}, r] \end{cases}$$

that deliver the same value of the cost. These trajectories have discontinuous derivative at the points r_1 and r_2 which leads to a contradiction.

Example 3.1 (Distance on a sphere: Columbus problem) Consider the problem of geodesics on a sphere again. Let us examine geodesics: They surely satisfy Weierstrass tests, but the Jacobi test is violated if the length of geodesics is larger than π times the radius of the sphere.

The argument that the solution to the problem of shortest distance on a sphere bifurcates was famously used by Columbus who argued that the shortest way to India might path through the West route. He was not able to prove or disprove his conjecture because he bumped into American continent discovering New World for better and for worst.