Variation of the domain

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1 Variation of the domains: Setting

Here, we derive the necessary conditions of optimality for the domain Ω where the Lagrangian is defined. The one-dimensional analog of this problem is the variation of the interval that leads to the transversality conditions, see Section (**??**). First, we derive optimality condition for the isoperimetric and related problem, where the Lagrangian depends solemnly on the domain itself (on volume, inertia moment, perimeter, etc.). Then, we consider a general problem for Lagrangian dependent on a minimizer inside the domain.

First we discuss the two-dimensional problem. Consider a region $\mathcal{O} \supset R_2$ ($\mathcal{O}$ can coincide with R_2) and a set of admissible domains Ω with the twice differentiable boundaryγ that belongs to the interior of $\mathcal{O}$. Consider also a differentiable minimizer $u(x)$, $x \in \mathcal{O}$, and a twice differentiable Lagrangian $F(x, u, \nabla u)$. Consider the variational problem (Mayer-Bolza problem).

$$J(\Omega) = \min_{u(x)} \left\{ \int_\Omega F(x, u, \nabla u) dx + \int_\gamma f(x, u) ds \right\}, \quad x \in \Omega \tag{1}$$

that contains the bulk and the boundary integrals. The objective functional $J(\Omega)$ depends on the domain Ω.

The domain optimization problem asks for a domain Ω that minimizes $J(\Omega)$ We come to a variational problem for the variable domain

$$I = \min_{\Omega \subset \mathcal{O}} J(\Omega). \tag{2}$$

Here, we study stationarity condition for optimal Ω. In order to define variation of the domain, assume that functions $F(x, u, \nabla u)$ and $f(x, u)$ are defined everywhere in a larger region $\mathcal{O}$.

Variation of the boundary We describe the boundary variation $\delta\gamma$ and related variation $\delta\Omega$. Consider an admissible domain $\Omega \cup \delta\Omega$ that differs from Ω by an infinitely thin domain $\delta\Omega$ with a twice differentiable boundary $\gamma \cup \Gamma$, as on Figure 1.

Figure 1: Variation of the domain

Denote the new part of the boundary of this domain by $\Gamma(s)$. In other words, we consider the added/subtracted domain $\delta\Omega$ as an infinitesimally thin strip of the width $\delta\eta$ and the boundary $\gamma \cup \Gamma$. As usual, we compute the variation of the objective

$$\delta I = J(\Omega \cup \delta\Omega) - J(\Omega)$$

Figure 2: Infinitesimal variation of the boundary of the domain

and analyze the stationarity condition $\delta I = 0$ which provides an additional boundary condition at the unknown optimal boundary. This way, we formulate the *free boundary problem*: a boundary value problem in an unknown domain but with an additional condition for this domain.

Remark 1.1 The considered variation does not change the topology of the domain: No new components of the boundary were added.

We start with establishing the coordinate system in proximity of $\partial\Omega$, particularly, the correspondence between points of the stationary boundary γ and the varied close-by boundary Γ. Define normal $n = n(s)$ to each point γ; the normal of a twice-differentiable curve is a continuous function of s. Introduce the following coordinates in an infinitesimal neighborhood of γ: a distance s along the curve and the (infinitesimal) distance η along the normal $n(s)$. Particularly, each point of γ is represented by $(s, 0)$. The admissible close-by curve Γ is represented as $\Gamma: \ \{s, \delta\eta(s)\}$ where $\delta\eta$ is an infinitesimal distance along the normal. Notice that $\delta\eta(s)$ may have any sign, the variation can go inside or outside of Ω.

A point in an infinitesimal strip $\delta\Omega$ can be identified by two coordinates: The distance s along the boundary line γ and the distance $z\delta\eta$ from this line computed along the normal, where $z \in [0, 1]$. Particularly, points at γ correspond to $z = 0$ and points at Γ – to $z = 1$.

We consider "thin and small" variations assuming that

$$\delta\eta = 0 \ \text{ if } s \notin (s_0,\, s_0 + \epsilon), \quad \delta\eta \ll \epsilon \tag{3}$$

If the curvature k of γ is not zero, the length of the arch varies with the normal. The infinitesimal length dS of the part of the curve along Γ is related to the infinitesimal length ds along γ as

$$dS = ds(1 + k\delta\eta) + o(\delta\eta) \tag{4}$$

This formula is known in differential geometry as ???? , see [] (see for example **??**) Figure 3 illustrates it:

Figure 3: The variation of the differential of a curve due to its curvature

Remark 1.2 Here, we assume that Ω belongs to an open set $\mathcal{O}$. If this set is closed, the consideration must be supplemented by the inequalities (see Section ??). The Lagrange multipliers conventionally account additional constraints on Ω such as the prescription of its volume or perimeter.

Remark 1.3 The variation of singular points of the domain where the normal is not defined must be considered separately, see the examples below.

2 Geometric Lagrangian

2.1 Stationarity conditions

First we show how the bulk and boundary integrals in (1) vary due to the variation of the boundary in the case when the integrands are independent of u and are continuous bounded functions of x only, $F = F(x)$ and $f = f(x)$. In other words, consider the variational problem

$$I = \min_{\Omega \subset \mathcal{O}} J(\Omega), \quad J(\Omega) = \int_\Omega F(x)dx + \int_\gamma f(s)ds, \quad x \in \Omega. \tag{5}$$

that depends only on the shape of the region Ω.

Increment of the functional due to variation of the integral over Ω The integral of a continuous function $F(x)$ over an infinitely thin domain is estimated through the boundary integral

$$J_{\delta\Omega} = \int_{\delta\Omega} F(x)dx = \int_\gamma F(s)\delta\eta(s)ds + o\|\delta\eta(s)\|. \tag{6}$$

To obtain this formula, we compute the integral $J_{\delta\Omega}$ as the repeated integral

$$J_{\delta\Omega} = \int_\gamma \delta\eta \left(\int_0^1 F(s, z\delta\eta)dz \right) ds \tag{7}$$

integrating first along the normal n and then over the arch γ. The inner integral is estimated using the continuity of F and the smallness of $\delta\eta$, as

$$\int_0^1 F(s, z\delta\eta)dz = \delta\eta\, F(s, 0) + o(\delta\eta)$$

and the formula (6) is obtained by computing the first-order term. Here, we denote $F(s, 0)$ as $F(s)$.

Variation of the boundary integral The value of the differentiable function $f(x)$ at the point $S \in \Gamma$ is expressed as

$$f(S) = f(s) + \delta\eta \frac{\partial f}{\partial x} n + o(|\delta\eta|) \tag{8}$$

where $\frac{\partial f}{\partial x} n = n^T \nabla f$ is the normal derivative of f.

The variation of an integral of a function $f(x)$ can be computed using (4) and (8) and rounding to $|\delta\eta|$

$$\begin{aligned} \int_\Gamma f(S) dS - \int_\gamma f(s) ds &= \int_\gamma \left[\left(f(s) + \delta\eta \frac{\partial f}{\partial x} n \right) (1 + k\delta\eta) - f(s)) \right] ds \\ &= \int_\gamma \left(k f(s) + \frac{\partial f(s)}{\partial n} \right) \delta\eta \, ds + o(\delta\eta). \end{aligned} \tag{9}$$

Stationarity conditions Adding together the two increments (6) and (9), we find that the increment δI in (5) is equal to

$$\delta I = \int_\gamma \left(F(s) + k f(s) + \frac{\partial f(s)}{\partial n} \right) \delta\eta \, ds.$$

As usual, we use the arbitrariness of $\delta\eta$ to obtain the stationary condition

$$F(s) + \left(k + \frac{\partial}{\partial n} \right) f(s) = 0 \quad \forall s \in \gamma \tag{10}$$

for this simplest case (5) where the Lagrangian is a fixed function of x.

Example 2.1 (Isoperimetric problem) Consider the isoperimetric problem: Maximize the area of Ω keeping its perimeter equal to one:

$$\max_\Omega \int_\Omega dx \quad \text{subject to} \quad \int_\gamma ds = 1 \tag{11}$$

or

$$\max_\Omega \left\{ \int_\Omega dx + \Lambda \int_\gamma ds \right\}$$

where Λ is the Lagrange multiplier by the isoperimetric constraint. Here we put

$$F = 1, \quad f = \Lambda$$

The necessary condition (10) gives $1 + k\Lambda = 0$

$$k = -\frac{1}{\Lambda} = \text{constant} \tag{12}$$

at the optimal boundary. The unknown boundary is a circle (or a part of a circle).

Remark 2.1 (Comparison with symmetrization technique) The isoperimetric problem can be solved by the symmetrization method, see Sectionsymm. Contrary to the symmetrization method, the variational technique applies to a broader range of problems. Here, for instance, the necessary conditions are applicable for the case if a part of the boundary of Ω is fixed. The unfixed boundary components are the arcs of constant curvature, joined by the fixed boundary components. The curvature of the unfixed components is constant everywhere and can be found from the isoperimetric condition.

On the other hand, the obtained condition does not prove that the shape with constant curvature deliver the global minimum of the isoperimetric problem because only close-by trajectories were compared, while the symmetrization method guarantees the global minimum but cannot withstand the additional constraints.

Example 2.2 (A domain with an extremal moment and fixed perimeter) Consider the problem of the symmetric domain with the maximal moment of inertia M

$$M = \int_\Omega x^2 dx \tag{13}$$

that passes through two given points $(-a, 0)$ and $(a, 0)$ and has a fixed perimeter. We set $F = x^2$, $f = \lambda$, where λ is the Lagrange multiplier. The optimality condition is

$$\lambda k = x^2.$$

In a Cartesian coordinates, the boundary of the domain consists of a pair of curves $\pm y(x)$, the curvature is given by the known expression $k = \frac{y''}{1+y'^2}^{\frac{3}{2}}$, and the optimality condition leads to the equation

$$y'' = \frac{1}{\lambda} x^2 (1 + y'^2)^{\frac{3}{2}}$$

Separating the variables and using symmetry $(y'(0) = 0)$, we obtain

$$y'(x) = \frac{x^3 \lambda}{\sqrt{9 - x^6 \lambda^2}}$$

Next , we find $y(x)$ and calculate λ using condition (13). The graph of the optimal curve is shown in Figure **??**.

Notice that the curve is smooth everywhere except the points $(-a, 0)$ and $(a, 0)$ where its values are prescribed.

2.2 Geometric applications

2.3 Cluster of domains with minimal boundary length

Boundary components Assume that two finite domains Ω_1 and Ω_2 of given areas A_1 and A_2, respectively, have a common component γ_{12} of the boundary. Consider the problem of the shapes of these domains that minimizes the total length of the boundary.

Figure 4: Optimal angle between normals in the singular point
Left: The angle between any two normals is less than 180°
Right: The angle between any two normals equals 120°

The domain $\Omega = \Omega_1 \cup \ldots \Omega_N$ has outer and inner boundaries. The optimality conditions for variation of the outer boundary is similar to the condition (12) in the isoperimetric problem:

$$\delta\eta_i : \quad k_i + \Lambda_i = 0, \quad i = 1, 2.$$

The conditions indicate that the outer bound is composed of two circular arcs with radii $R_1 = 1/k_1$ and $R_2 = 1/k_2$, respectively.

Let us call γ_{12} an inner component of the boundary that separates Ω_1 and Ω_2. The variation of γ_{12} of the boundary increases the volume of one of the neighbor and decreases the volume of the second one by the same amount. The resulting condition is:

$$\delta\eta_{12} : \quad k_{12} + \Lambda_1 - \Lambda_2 = 0$$

This condition implies that the dividing lines are also circular, with the radius (a reciprocal to the curvature) R_{ij} equal to

$$\frac{1}{R_{12}} = \frac{1}{R_1} - \frac{1}{R_2} \tag{14}$$

In particular:

– The boundary between two domains of equal areas is straight.

– Smaller domain remains convex when it is joined with the larger one, and the larger is of crescent shape.

- If one of the domains is infinitely large, the smaller domain is a symmetric lens.

Singular points To complete the consideration, we determine the angle at the point s_0 where three components of the boundary meet. This problem requires special consideration because the boundary curve is not smooth, the normal at the point s_0 is not defined, so the standard variations are not possible.

First, it is easy to show that the components stay continuously differentiable until they meet. Assume that a boundary component has an angular point s_0 and the normal is discontinuous there, denote the angle ϕ. Consider two points infinitely close points s_1 and s_2 at the boundary at different sides of s_0 such that $|s_1 - s_0| = |s_2 - s_0| = \epsilon$ and a triangle $s_0 s_1 s_2$. Assume that the varied boundary replaces curve $s_2 s_0 s_1$ with the straight line $s_1 s_2$ of the length $|s_1 s_2| = 2\epsilon\cos\phi$. The variation δl of boundary lengths is negative $\delta I = 2\epsilon(\cos\phi - 1)$ (which is evident from triangle inequality) and is of the order ϵ, the variation of the area $\delta A = \epsilon^2(\cos\phi - 1)$ is of the order of ϵ^2 and therefore is neglected in the linear part of increment. We conclude that there cannot be kinks at an optimal boundary.

Consider now a meeting point s_0 of intersection of three boundary components. Again, choose three points s_1, s_2 and s_3 at the corresponding branches of the boundaries in ϵ-neighborhood of s_0, so that $s_i \neq s_0$. s_0 is optimal, if

$$L(s_0) = \sum_{i=1}^{3} |s_i - s_0|$$

is minimal. Differentiation with respect to s_0 and using the equality $\frac{d}{dx}|x| = \frac{x}{|x|}$, we obtain equation for an optimal s_0,

$$\frac{dL(s_0)}{ds_0} = \sum_{i=1}^{3} \tau_i = 0, \quad \tau_i = \frac{s_i - s_0}{|s_i - s_0|}.$$

where τ_i are unit vectors codirected with $s_i - s_0$. The stationarity stays that the sum of these three unit vectors is zero, therefore these vectors are directed at $120°$ to each other.

Again, the increment of the variation of lengths is of the order of ϵ, and the increment of areas is of the order of ϵ^2 and is neglected in the linear term. We conclude that

Theorem 2.1 Optimal boundary components meet at the angle $120°$ to each other.

Definition of parameters We found that the boundaries of two-domain configuration consist of three arches of circles which radii are connected by the condition (14); that intersect in two symmetric points, the angle between the arches equal to $120°$. To draw the configuration, it is enough to find radii R_1, R_2 and R_{12}, and angles θ_1, θ_2 of the arches. To find these five parameters we have two equations that express the conditions of intersections

$$R_1 \sin\theta_1 = R_2 \sin\theta_2 = R_{12} \sin\theta_{12}$$

equation (14), and two equations that fix the areas of the domains

$$A_1 = R_1(\pi - \theta_1) + \frac{1}{2}R_1^2 \sin\theta_1 \cos\theta_1 + R_{12}\theta_{12} - \frac{1}{2}R_{12}^2 \sin\theta_{12}\cos\theta_{12}$$

and

$$A_2 = R_2(\pi - \theta_2) + \frac{1}{2}R_2^2 \sin\theta_2 \cos\theta_2 - R_{12}\theta_{12} + \frac{1}{2}R_{12}^2 \sin\theta_{12}\cos\theta_{12}$$

and the condition

$$\theta_1 - \theta_{12} = \theta_2 - \theta_{12} = 2\pi/3$$

too many conditions!

Summing up, the formulate the theorem:

Theorem 2.2 The optimal boundary consists of three circular arches that meet at two symmetric points at 120°; the curvatures of the arches are related as $k_{12} = k_1 - k_2$.

For instance, if the areas are very distinct, $A_1 \ll A_2$, the smaller area is a lens made of the circles of the close-by radii circle and the larger area is a circle without a lens-shaped area.

Multiple domains A natural generalization of the previous problem is the problem of N separated domains Ω_i, $i = 1, \ldots, N$ of given areas A_i with the minimal length of separating boundary. The domain $\Omega = \Omega_1 \cup \ldots \Omega_N$ has outer and inner boundaries. The optimality conditions for variation of the outer boundary are similar to the condition (**??**) in the isoperimetric problem:

$$\delta\eta_i : \quad k_i + \Lambda_i = 0$$

Let us find the conditions of optimality of an inner component of the boundary. Suppose that two domains Ω_i and Ω_j are neighbors. The variation of their common boundary ∂_{ij} results in the condition

$$\delta\eta_{ij} : \quad k_{ij} + \Lambda_i - \Lambda_j = 0$$

which implies that the dividing lines are also circular, with the radius (a reciprocal to the curvature) R_{ij} equal to

$$\frac{1}{R_{ij}} = \frac{1}{R_i} - \frac{1}{R_j}$$

If the number of domains with different areas is larger of or equal to four, the question arises what domains should be placed inside the configuration and do not have an outer boundary. More generally, the question is which domains should be neighbors. This problem requires combinatoric methods since it is needed to compare several not close-by configurations. We do not discuss the problem in this text but encourage the reader to try to solve it.

Example 2.3 An infinite system of domains of equal areas forms a honeycomb structure. Bees know variational calculus! Indeed, all the inner boundaries are straight due to symmetry, and all angles, where the boundaries meet, are equal to 120°.

Optimal shapes in a bounded domain When a shape touches the boundary of the domain, it meets this boundary at a right angle. Indeed, the angle which corresponds to minimal length of γ to the boundary is a right angle. The rest of the consideration is as before.

Natural cracks in Badlands Photo! Observe the cracks meeting either at 120^o or at 90^o Discuss the history of the development of the cracks.

2.4 Three-dimensional problems: Minimal surface and shape of bubbles

Three-dimensional problem The consideration of the three-dimensional case is similar, but the formula for the variation of the boundary arch is replaced by the formula for the variation of the boundary surface element:

$$dS = ds(1 + k_1\delta\eta)(1 + k_2\delta\eta) = ds(1 + (k_1 + k_2)\delta\eta) = o(\delta\eta) \tag{15}$$

where k_1 and k_2 are the main curvatures on the boundary surface. The calculation of the stationary condition is performed as before. The necessary condition:

$$F(s) + \left(k_1 + k_2 + \frac{\partial}{\partial n}\right) f = 0 \quad \text{on } \delta\Omega \tag{16}$$

differs from the two-dimensional case by replacing the curvature k of the boundary line with the mean curvature $k_1 + k_2$ of the boundary surface.

The simplest problem is the minimal surface problem: Find a surface of the minimal area that is attached to a given contour. Here,

$$F = 0 \quad f = 1$$

Remark 2.2 The surface may be not closed, but this is of no importance since we consider the only local variation of the "boundary" surface.

The stationarity condition

$$k_1 + k_2 = 0$$

shows that the minimal surface has zero mean curvature.

Bubble The bubble problem is the three-dimensional analog of the isoperimetric problem: Find a domain of maximal volume enclosed in the surface of the fixed area. Here

$$F = 1 \quad f = \Lambda$$

and the optimality condition

$$k_1 + k_2 = \Lambda$$

stays that the mean curvature is constant. A sphere is an obvious solution. Beside it, there are many other shapes: a circular cylinder, for example. One may experiment with an air-balloon to find these shapes. The problem is formulated as a partial differential equation that expressed the constancy of the mean curvature of the surface with the boundary conditions that correspond to the requirements that the surface passes through a given contour. For references, see [].

Cluster of bubbles A problem of a minimal total boundary area of a cluster of several spacial domains with given volumes is considered similarly to the two-dimensional case. Physically, this is the problem of a cluster of bubbles.

The cases of two, three, and four equal bubbles can be handed out analytically, using the obvious symmetries. In the general case, the outer and interior surfaces of the cluster satisfy the following conditions:

Theorem 2.3 (Bubble theorem) The following statements describe the cluster of the bubbles:

(1) The outer boundary of the cluster consists of the surface with piece-wise constant mean curvature.

(2) Any two volumes (bubbles) are divided by a surface of constant mean curvature.

(3) Any three bubbles are divided by curves formed by an intersection of three mean curvature surfaces with normals that meet at the angles $2\pi/3$ independently of the volumes of bubbles.

(4) Any four bubbles meet at the points where four mean curvature surfaces meet. The normals to these surfaces at the meeting point meet at the angles $2\pi/3$ as the planes meet in the center of a symmetric tetragon that passes through the sides of this tetragon. The angles between the meeting planes are fixed independently of the volumes of bubbles.

Find a reference

Hints for the proof

(1) Apply the necessary conditions on a free boundary

(2) Apply the necessary conditions on a dividing boundary

(3) Consider an infinitesimally thin cylinder abound the line of intersection.

(4) Consider an infinitesimal sphere or tetragon around the point of intersection.

3 General form of Lagrangian

3.1 Formulas for the increment

Here we consider the general form of the functional (2) assuming that the integrants depend on a minimizer $u(x)$, $x \in \Omega$ and the domain Ω is varied. We derive an additional boundary condition at the optimal domain which serves for finding this domain.

The variation of the functional in (2) is computed as the difference

$$\begin{aligned}
\delta J_n &= \delta J_1 + \delta J_2 \\
\delta J_1 &= \int_{\Omega \cup \delta\Omega} F(x, u + \delta u, \nabla(u + \delta u))dx - \int_{\Omega} F(x, u, \nabla u)dx \\
\delta J_2 &= \int_{\Gamma} f(x, u + \delta u)dS - \int_{\gamma} f(x, u)ds \qquad (17)
\end{aligned}$$

between the cost of the Lagrangians of an admissible solution in an admissible domain and of the extremal solution in the optimal domain.

3.2 Variation of the bulk integral

First, we work out the increment δJ_1 of bulk integrals. We rewrite it as

$$\begin{aligned}
\delta J_1 = &\int_{\Omega \cup \delta\Omega} F_u \delta u \, dx + \int_{\delta\Omega} F(x, u, \nabla u)dx \\
&+ \int_{\Gamma} n \cdot \frac{\partial F}{\partial \nabla u} \delta u(\Gamma) \, dS + o(\|\delta u\|, \|\delta\eta\|) \qquad (18)
\end{aligned}$$

where

$$F_u = -\nabla \cdot \frac{\partial F}{\partial \nabla u} + \frac{\partial F}{\partial u}$$

To derive this formula, we add and subtract the integral over $\delta\Omega$ of $F(x, u, \nabla u)$ and compute the fist variation δJ_1 with respect to δu in $\Omega \cup \delta\Omega$.

The variation δu at $\Gamma = \gamma + \delta\eta$ that is a sum of the variation of minimizer $\delta u(\gamma)$ at stationary boundary γ and shift $\frac{\partial F}{\partial u}\delta\eta$ of the minimizer due to variation of the boundary,

$$\delta u(\Gamma) = \delta u(\gamma) + \frac{\partial u}{\partial n} \delta\eta$$

In other words, variation $\delta u(\gamma)$ is expressed as the linear combination of two free variations $\delta u(\Gamma)$ and $\delta\eta$:

$$\delta u(\gamma) = \delta u(\Gamma) - \frac{\partial u}{\partial n} \delta\eta$$

Integral over $\partial\Omega$ in the right-hand side of (18) is estimated as

$$\int_{\Omega \cup \delta\Omega} F(x, u, \nabla u)dx = \int_{\gamma} F(x, u, \nabla u)\delta\eta dx + o(\|\delta\eta\|)$$

It remains to substitute these expressions into (18) and group the terms:

$$\begin{aligned}
\delta J_1 &= \int_{\Omega} S_u \delta u \, dx + \int_{\gamma} A_u \delta u \, ds + \int_{\gamma} A_\eta \delta\eta \, ds \qquad (19) \\
S_u &= \frac{\partial F}{\partial u} - \nabla \cdot \frac{\partial F}{\partial \nabla u}, \quad A_u = n \cdot \frac{\partial F}{\partial \nabla u} \quad A_\eta = F - \left(n \cdot \frac{\partial F}{\partial \nabla u}\right) \frac{\partial u}{\partial n} \qquad (20)
\end{aligned}$$

The first two terms in (19) are standard conditions of stationarity of u; the last term expressed the stationarity of Ω.

Example 3.1 Let F by

$$F = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right] + q\,u$$

We compute

$$S_u = \nabla^2 u - q, \qquad n = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \quad A_u = \frac{\partial u}{\partial x_1}\cos\theta + \frac{\partial u}{\partial x_2}\sin\theta,$$

$$\left(n \cdot \frac{\partial F}{\partial \nabla u}\right)\frac{\partial u}{\partial n} = \left(\frac{\partial u}{\partial x_1}\cos\theta + \frac{\partial u}{\partial x_2}\sin\theta\right)^2 = \left(\frac{\partial u}{\partial n}\right)^2$$

Using invariant of ∇^2 to rotation,

$$\nabla^2 = \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 = \left(\frac{\partial u}{\partial n}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2$$

we obtain

$$A_\eta = \frac{1}{2}\left(\frac{\partial u}{\partial n}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial n}\right)^2 = \frac{1}{2}\left(\frac{\partial u}{\partial n}\right)^2 - \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2$$

or, in x_1, x_2 coordinates,

$$t = \begin{pmatrix}\sin\theta \\ -\cos\theta\end{pmatrix}, \quad A_\eta = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2\right]\cos 2\theta + \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_1}\sin 2\theta$$

Increment δJ_1 is analogous to the transversality condition in the one-dimensional problem, see Chapter ??? (**??**). The first term S_F is the accumulated value of the Lagrangian over the added domain $\delta\Omega$ and the second term $-n^T \frac{\partial F}{\partial \nabla u}\delta\eta$ is a shift of natural boundary conditions from the admissible boundary to the stationary boundary; this term is an analog of the term $-u' \frac{\partial F}{\partial u'}\delta x$ in the one-dimensional transversality condition.

If the boundary Lagrangian f is zero, the additional (transversality) boundary condition $A_\eta = 0$ holds at the unknown boundary γ and serves to find this boundary.

Example 3.2 (Vibrating membrane of minimal frequency) Consider the problem of the vibrating membrane of a fixed area with minimal first frequency,

$$\min_\Omega \min_u \int_\Omega F(u, \nabla u)dx, \quad \text{if} \int_\Omega dx = A.$$

The extended Lagrangian $F_\Lambda =$ is $F_\Lambda = \frac{1}{2}\nabla^2 u - \frac{1}{2}c^2u^2 + \Lambda$ where Λ is Lagrange multiplier by the condition fixing the area, u is the deflection, and ω is the frequency. Assume that the membrane is fixed on the boundary γ,

$$u(\gamma) = 0. \tag{21}$$

The stationarity condition with respect to u is expressed through the Euler equation

$$\nabla \cdot \frac{\partial F_\Lambda}{\partial \nabla u} - \frac{\partial F_\Lambda}{\partial u} = \nabla^2 u + c^2 u = 0,$$

The stationarity of the shape of the domain (24) is computed as follows: In this problem, $f = 0$, and $\frac{\partial u}{\partial s} = 0$ on γ. It is convenient to compute R_F in local coordinates – the tangent t and the normal n to the boundary. F is invariant to the orientation of the coordinate axis. In n, t coordinates, F has a form

$$F = \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial n}\right)^2 - c^2u^2\right] + \Lambda \qquad \left(\frac{\partial F}{\partial \nabla u}\cdot n\right)\frac{\partial u}{\partial n} = \left(\frac{\partial u}{\partial n}\right)^2$$

On the boundary γ, $u = 0$ and therefore $\frac{\partial u}{\partial t} = 0$. We have

$$R_F = F(s) = -\frac{1}{2}\left(\frac{\partial u}{\partial n}\right)^2 + \Lambda = 0 \quad \text{or} \quad \frac{\partial u}{\partial n} = \sqrt{2\Lambda} = \text{ constant} \quad \text{on } \gamma \quad (22)$$

This condition is used to find the unknown boundary γ. One easily guesses that the circular membrane has an optimal shape.

3.3 First variation of the boundary integral

To compute the curve integral J_2, we expand $f(X, u(X))|_{X\in\Gamma}$ as follows:

$$f(X, u(X))|_{X\in\Gamma} = f(x)|_{x\in\gamma} + (\nabla f)^T n\, \delta\eta + \frac{\partial f}{\partial u}\delta u + o(\|\eta\|)$$

where n is the normal, and

$$(\nabla f(x,u))^T n = \left(\frac{\partial f}{\partial x}\right)^T n + \frac{\partial f}{\partial u}\frac{\partial u}{\partial n}$$

Example 3.3 Let $f(x, u)$ and n be

$$f(x,u) = \frac{1}{2}(x_1^2 + x_2^2)u^2, \quad n = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

We compute

$$\left(\frac{\partial f}{\partial x}\right)^T n = \frac{\partial f}{\partial x_1}\cos\theta + \frac{\partial f}{\partial x_2}\sin\theta = (x_1\cos\theta + x_2\sin\theta)u^2$$

and

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x_1}\cos\theta + \frac{\partial u}{\partial x_2}\sin\theta, \quad \frac{\partial f}{\partial u}\frac{\partial u}{\partial n} = u(x_1^2 + x_2^2)\left(\frac{\partial u}{\partial x_1}\cos\theta + \frac{\partial u}{\partial x_2}\sin\theta\right)$$

and combine the terms

$$(\nabla f(x,u))^T n = (x_1\cos\theta + x_2\sin\theta)u^2 + (x_1\cos\theta + x_2\sin\theta)u^2$$

We take into account the variation of the boundary arch (4) ($dS = ds(1 + k\delta\eta)$) and obtain

$$\delta J_2 = \int_\gamma \left[f(x,u) + \frac{\partial f}{\partial u}\delta u + (\nabla f(x,u))^T n\delta\eta \right] (1 + k\delta\eta) ds - \int_\gamma f(x,u) ds$$

Rounding to $\delta\eta$, we have

$$\delta J_2 = \int_\Gamma \left(\left[k\, f + (\nabla f(x,u))^T n \right] \delta\eta + \frac{\partial f}{\partial u}\delta u \right) ds$$

The first term in parenthesis accounts for increase of the length of varied curve Γ, the next term shows the shift of the $f(s)$ due to variation $\delta\eta$ of normal to γ, and the last term is the variation due to variation of u.

Example 3.4 Assume that the problem asks for the shortest path between two points. We set $F = 0$, $f = 1$. The optimal boundary corresponds to the condition $\delta J_2 = 0$ for all $\delta\eta$, which leads to $k = 0$. The optimal path is a straight line, as expected.

Stationary conditions Adding δJ_1 and δJ_2, we finally obtain the stationarity conditions:

The variations with respect to u returns the familiar expressions: F_u is the left-hand-side of the Euler equation

$$\delta u(x): \quad F_u = \nabla \cdot \frac{\partial F}{\partial \nabla u} - \frac{\partial F}{\partial u} = 0 \quad \text{in } \Omega,$$

natural boundary condition,

$$\delta u(s): \quad \Phi_u \delta u(s) = 0, \quad \Phi_u = \frac{\partial F}{\partial \nabla u}\boldsymbol{n} - \frac{\partial f}{\partial u} = 0 \quad \text{on } \gamma, \tag{23}$$

and the stationarity of the domain:

$$R_F \delta\eta = 0, \quad R_F = F - \left(\frac{\partial F}{\partial \nabla u} \cdot n \right) \frac{\partial u}{\partial n} + (\nabla f)^T n + k\, f. \tag{24}$$

If the main boundary condition $u = u_0$ is prescribed at a boundary component $\partial_1\Omega$, then boundary integral independent of u, $f = f(x)$, and $(\nabla f)^T n = \frac{\partial f}{\partial n}$; the conditions on the optimal boundary become

$$u = u_0, \quad F - \left(\frac{\partial F}{\partial \nabla u} \cdot n \right) \frac{\partial u}{\partial n} + \frac{\partial f}{\partial n} = 0 \quad x \in \gamma. \tag{25}$$

Together, they determine the unknown boundary and provide a boundary condition of u.

If no boundary condition is prescribed (free boundary, or variational boundary conditions), then $\Phi_u = 0$ due to (23), and the last condition is simplified to

$$R_F = F + \frac{\partial f}{\partial x} n + k\, f = 0, \quad x \in \gamma. \tag{26}$$

They determine an unknown boundary and variational boundary conditions.

Remark 3.1 The conditions $\Phi_u = 0$ and $R_u = 0$ in the last condition depend on ∇u which we represent in local coordinates as $\nabla u = \frac{\partial u}{\partial n}\boldsymbol{n} + \frac{\partial u}{\partial s}\boldsymbol{t}$. Generally, this leads to two differential equations

$$\Phi_u = \Psi_1 \left(\frac{\partial u(s)}{\partial s}, u(s), \frac{\partial u(s)}{\partial n} \right) = 0$$

and

$$R_u = \Psi_2 \left(\frac{\partial u(s)}{\partial s}, u(s), \frac{\partial u(s)}{\partial n} \right) = 0$$

along the boundary. In these equations, $u(s)$ and $\frac{\partial u(s)}{\partial n}$ are independent functions.

Remark 3.2 (Three-dimensional problem) In the three-dimensional case, the formula for the variation stays the same except the curvature k is replaces by the mean curvature $k_1 + k_2$. The optimality condition (26) becomes :

$$\delta\eta \left(F(x, u, \nabla u) + (\nabla f)^T n + (k_1 + k_2) f \right) = 0$$

Here, we use formula (**??**) for the variation of boundary element.

Conditions on free boundary In many physical applications, $F = F(u, \nabla u)$ stays for the bulk energy, and $f = f(u)$ stays for the surface energy. We also assume that both forms of energy does not explicitly depend on the position x,

$$F = F(u, \nabla u), \quad f = f(u)$$

In this problem, the natural boundary conditions are imposed on the unknown component of the boundary.

Due to the listed conditions, the optimality condition on the unknown moving boundary simplifies and becomes:

$$F(u, \nabla u) - (k_1 + k_2) f(u) = 0 \tag{27}$$

We see that the mean curvature of the optimal boundary surface is the proportionality coefficient between the bulk and surface energies.

In particular, the constant value of f corresponds to the prescription of the surface area a three-dimensional body. In this case, the bulk energy density on the boundary is proportional to its mean curvature.

3.4 Optimal interior boundary

Similar consideration is applicable to the problem of optimal position of a variable boundary γ dividing two adjacent volumes Ω_1 and Ω_2. The problem is to minimize the the functional

$$I(\Omega_1, \Omega_2), = \int_{\Omega_1} F_1(u, \nabla u) dx + \int_{\Omega_2} F_2(u, \nabla u) dx + \int_\gamma f(u) dx \tag{28}$$

The necessary optimality condition (Weierstrass-Erdmann condition) on the boundary γ is

$$n^T \frac{\partial}{\partial \nabla u}[F(u, \nabla u)]_-^+ + \frac{\partial f(u)}{\partial u} = 0 \quad \text{on } \gamma$$

This is the condition on any boundary between two adjacent subdaomains volumes, obtained by the variation of u.

The optimal position of the boundary that minimizes (Ω_1, Ω_2) with respect interior boundary γ

$$I_0 = \min_{\gamma} I(\Omega_1, \Omega_2),$$

satisfies, in addition, the transversality condition:

$$[F(u, \nabla u)]_-^+ + k\, f(u) = 0 \quad \text{on } \gamma \tag{29}$$

Here, $[F]_-^+ = F_2 - F_1$ denotes the jump: The difference in the values of the function F at both sides of γ. Notice that the solution u and its tangential derivative $\frac{\partial u}{\partial t}$ are continuous at the boundary, but the normal derivative $\frac{\partial u}{\partial n}$ is, generally, discontinuous.

Example 3.5 (Optimal boundary between two conducting domains) Consider the domain occupied by two linearly conducting materials with vonductivities κ_1 and κ_2. The conductivity is described by the variational problem (28) where

$$W_1 = \frac{1}{2}\kappa_1(\nabla u)^2, \quad W_2 = \frac{1}{2}\kappa_2(\nabla u)^2,$$

u is the potential (say, temperature).

The Weierstrass-Erdmann condition

$$\kappa_1 \left.\frac{\partial u}{\partial n}\right|_{x \in \Omega_1} = \kappa_2 \left.\frac{\partial u}{\partial n}\right|_{x \in \Omega_2}$$

expresses the continuity of the normal current $j = \kappa \nabla u$. It is satisfied at any interior boundary. The optimal boundary that minimizes the total energy I satisfies the transversatily condition(29). Here $f = 0$, and the condition is $W_1 - W_2 = 0$, or

$$\kappa_1(\nabla u)^2 - \frac{1}{2}\kappa_2(\nabla u)^2 = 0$$

Using Weierstrass-Erdmann condition and continuity of the tangent derivative $\left.\frac{\partial u}{\partial t}\right|_-^+ = 0$, we obtain the condition

$$(\kappa_1 - \kappa_2) \left.\frac{\partial u}{\partial t}\right|_{x \in \gamma} = 0$$

which states that $\frac{\partial u}{\partial t} = 0$ on γ and therefore gradient is co-directed with the normal n to the dividing line, $\nabla u = \alpha n$. The optimal boundary between the subdomains is co-directed with the gradient of the minimizer u.

3.5 Optimal Shape of Conducting Domains

Next problems ask for the shape of the domain of extremal resistivity. Consider a domain Ω with the boundary $\gamma = \gamma_0 \cup \gamma_1 \cup \gamma_i$ divided into three components. Assume that the domain is filled with an isotropic material with unit conductivity. Assume that the potentials $u = 0$ and $u = 1$ are applied to the two components γ_0 and γ_1 of the boundary, respectively. The supplementary component γ_i of the boundary is insolated,

$$u = 0 \quad \text{on } \gamma_0, \quad u = 1 \quad \text{on } \gamma_1, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \gamma_i, \tag{30}$$

The resistivity is defined as the total conducting energy of the domain. Indeed, the total normal current $j = \frac{\partial u}{\partial n}$ through the γ_1 component of the domain is equal to the total energy of it:

$$\int_{\gamma_1} \frac{\partial u}{\partial n} ds = \int_{\gamma} u \frac{\partial u}{\partial n} ds = \min_u \frac{1}{2} \int_{\Omega} (\nabla u)^2 \, dx$$

The first equality follows from the boundary conditions (30) and the second – from Green's formula combined with the stationary condition $\nabla^2 u = 0$ in Ω.

Let us formulate an optimization problem: Maximize the total current through the γ_1 or, equivalently, minimize the overall conductivity of the domain by varying its boundary. Additional geometrical restriction can be assigned.

Assume also that the area s of the plain domain Ω is fixed. Consider the conductivity problem in a domain of the fixed volume and a partly known boundary

$$\min_{\gamma_i} \left(\min_{u \text{ as in } (30)} \frac{1}{2} \int_{\Omega} (\nabla w)^2 dx + \lambda \int_{\Omega} dx \right)$$

where λ is the Lagrange multiplier. The potential u on the known components is prescribed and natural boundary conditions are satisfied on γ_i.

The augmented Lagrangian is

$$F = \frac{1}{2} (\nabla u)^2 + \lambda, \quad f = 0$$

and λ is the Lagrange multiplier. The Euler-Lagrange equation and natural boundary condition are

$$\nabla^2 u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \gamma_i$$

One more condition is needed to define the shape of unknown boundary component γ. This condition on the unknown boundary is

$$n^T \frac{\partial F}{\partial \nabla u} - F = \left(\frac{\partial u}{\partial n} \right)^2 - \left(\frac{\partial u}{\partial t} \right)^2 - \lambda = 0 \tag{31}$$

where t is the tangent to the boundary.

Example 3.6 (Conditions on optimal insulated boundary component) Assume that the shape of boundary components γ_0 and γ_1 is known, but the insulated component γ_i is movable. Combining the optimality condition (31) with the stationary condition $\frac{\partial u}{\partial n} = 0 \text{ on } \gamma_i$ we obtain the condition for the unknown boundary:

$$w_t = \sqrt{-\lambda} = \text{constant} \quad \text{on } \gamma$$

This condition tells that the optimal isolated boundary is also the current line: the current density along it is constant.

For example, a rectangular domain is optimal if the potentials on two opposite sides of it are constants.

Example 3.7 (Optimal boundary component with the given potential) Suppose now that the boundary component γ_0 where the main boundary condition $w = C$ is imposed should be found from the optimality requirements. Again, we use the condition (31) combining it with the prescribed boundary conditions in the form

$$\frac{\partial u}{\partial t} = 0 \quad \text{on } \gamma_0, \text{ and } \gamma_1$$

The optimality of this component is expressed as

$$w_n = \sqrt{\lambda},$$

This condition implies the constancy of the normal current on the optimal boundary where the potential is constant.

Example 3.8 (Optimal boundary of the domain with given perimeter) Consider the previous conductivity problem in a domain with the boundary of the fixed perimeter but arbitrary volume; this time

$$F = (\nabla w)^2, \quad f = \Lambda$$

where Λ is the Lagrange multiplier. The condition on the unknown boundary is

$$F + \Lambda k = 0 \quad \text{or } w_n^2 + w_t^2 + \Lambda k = 0$$

Combining it with the stationary condition

$$w_n = 0 \text{ on } \gamma$$

with respect to w, we obtain the condition

$$w_t^2 = \Lambda k.$$

which says that the square of the flux density along the boundary is proportional to its curvature.

Example 3.9 (Optimal shape of the membrane with minimal eigenfrequency) Consider the problem of the shape of a membrane of minimal eigenfrequency and given volume. Here

$$F = (\nabla w)^2 + c^2 w^2 + \Lambda, \quad f = 0$$

Here Λ is the Lagrange multiplier accounting for the volume constraint, w is the deflection of the membrane which is zero on its boundary,

$$w = 0 \quad \text{on } \gamma.$$

The tangent derivative among the boundary $\frac{\partial w}{\partial t}$ is zero as well. The optimality condition becomes

$$\frac{\partial w}{\partial n} = \text{constant},$$

it shows that the normal derivative of the deflection is constant along the boundary. One easily guesses that the optimal shape is the circle.