Vector Minimizers

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Contents

1 Lagrangians with Div and Curl

1.1 Div and Curl operators

Dealing with variational problems in physics, we meet the Lagrangians (energies) that depend on a vector potential and their linear differential forms. The common minimizers include curl and divergence: they are met in Maxwell equations, equations of hydrodynamics, transport equations, and other applications. The equations of solid mechanics deal with tensor arguments such as strain (deformation) or stress; these variables are also special combinations of the derivatives of corresponding potentials. For example, the strain is the symmetric part of the displacement gradient ∇u and depends on the sum $\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}$. In such problems, Lagrangian depends not on all partial derivatives $\nabla \eta$ of the corresponding vector potentials, but only on a special combination of them. The electromagnetic energy depends on the curl of the magnetic potential. The energy of a gas depends on the pressure which is a divergence of the speed. These forms of the energy lead to special forms of the Euler-Lagrange equations which we derive here considering the Lagrangians of the type $L(u, \nabla \cdot u, \nabla \times u)$, where u is a vector minimizer. In Chapter **??**, we generalize the technique to Lagrangians that depends on an arbitrary linear combination of partial derivatives of the vector potential.

Some formal identities Before deriving the equations, we explain the structure of the Curl and Divergence operators. Both of them are linear combinations of partial derivatives of a vector field. Consider a vector field $v = [v_1, v_2, v_3]^T$ in R^3 where v_i are differentiable function of x. The 3×3 matrix ∇v is a list of all partial derivatives

$$\nabla v = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}$$

The trace of this matrix is called the *divergence* of v

$$\nabla \cdot v = \text{ Tr}\, \nabla v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_1} + \frac{\partial v_3}{\partial x_1}$$

The antisymmetric part

$$\nabla^A v = \frac{1}{2}(\nabla v - (\nabla v)^T) = \begin{pmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_3 & -c_1 & 0 \end{pmatrix}$$

is defined by three nonzero entrances

$$c_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \quad c_2 = \frac{\partial v_1}{\partial x_1} - \frac{\partial v_3}{\partial x_1}, \quad c_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

These entrances form the vector $\nabla \times v = (c_1, c_2, c_3)$. The matrix is called the adjoint to $\nabla \times v$ (CHECK IT) matrix.

We mention the identities

$$\nabla \times \nabla \times u = \nabla^2 u - \nabla \cdot \nabla u \tag{1}$$

$$\nabla \times \nabla u = 0 \tag{2}$$

$$\nabla \cdot (\nabla \times u) = 0 \tag{3}$$

for the second-order differential operations, that is easy to check by straightforward calculation.

Helmholtz's decomposition theorem The Helmholtz's theorem states that any vector field v satisfyiing the conditions

$$|\nabla \cdot v| \to 0, \quad |\nabla \times v| \to 0, \quad \text{if } |x| \to \infty$$

can be presented as a sum of a solenoidal part $\nabla \times \Phi$ and irrotational part $\nabla \phi$ as

$$v = \nabla \times \Phi + \nabla \phi \tag{4}$$

where ϕ is a scalar and Φ - vector potential, that can be found as

$$\begin{aligned} \phi &= \frac{1}{4\pi} \int_{R^3} \frac{\nabla \cdot v}{|x-y|} dy \\ \Phi &= \frac{1}{4\pi} \int_{R^3} \frac{\nabla \times v}{|x-y|} dy \end{aligned}$$

Notice that the irrotational part depends on the divergence $\nabla \cdot v$ and the solenoidal - on the curl $\nabla \times v$ of v.

Divergence and curl operators often enter Lagrangians in variational problems related to various physical applications. In the next two sections, we derive Euler-Lagrange equations for such Lagrangians.

1.2 Lagrangian dependent of Divergence

Consider the Lagrangian $F(x, u, \nabla \cdot u)$ where $F(x, y, z)$ is the twice-differentiable function of two d-dimensional vectors x and y and a scalar z, and the differentiable boundary Lagrangian $f(s, u)$. Consider the variational problem

$$\min_u J, \quad J = \int_\Omega F(x, u, z) dx + \int_{\partial\Omega} F_\partial(s, u) ds, \quad z = \nabla \cdot u$$

where u is a vector minimizer, and $F_\partial(u)$ is a boundary Lagrangian.

To compute the stationarity of J, we rewrite the Lagrangian as a function of u and ∇u, using the representation $z = \nabla \cdot u = \operatorname{Tr} \nabla u$. We represent the Lagrangian as $F(x, u, z) = F(x, u, \operatorname{Tr} \nabla u)$ and apply the general formula (**??**). We compute, using (**??**)

$$\frac{\partial F}{\partial \nabla u} = \frac{\partial F}{\partial z} \frac{\partial z}{\partial \nabla u} = \frac{\partial F}{\partial z} I$$

Notice that the right-hand side contains a scalar factor $\frac{\partial F}{\partial z}$ and identity matrix I. The Euler-Lagrange equation becomes a vector-valued equation

$$-\nabla\left(\frac{\partial F}{\partial z}\right)+\frac{\partial F}{\partial u}=0$$

because

$$\nabla\cdot(a(x)I)=\nabla a(x)$$

The natural boundary conditions are computed using the equality

$$n^T\left(\frac{\partial F}{\partial z}I\right)=\frac{\partial F}{\partial z}n$$

General formula (**??**) for natural boundary conditions becomes

$$\frac{\partial F}{\partial z}n+\frac{\partial F_\partial}{\partial u}=0 \quad \text{on } \partial\Omega.$$

Notice that the first term is a vector codirected with the normal to the boundary, while the second is an arbitrary directed vector. Projecting this equality to the normal n and tangent(s) t to $\partial\Omega$, we obtain the system of boundary conditions

$$\begin{aligned}\frac{\partial F}{\partial z}&=n^T\frac{\partial F_\partial}{\partial u}\\ 0&=t^T\frac{\partial F_\partial}{\partial u}\end{aligned}$$

If $F_\partial=0$, the boundary conditions become $\frac{\partial F}{\partial z}n=0$. Although this is a vector condition, it is is fulfilled when a single condition

$$\frac{\partial F}{\partial z}=0$$

is satisfied. In this case, the stationarity conditions lead to an under-determined boundary value problem and the solution – the minimizer $u(x)$ is not unique.

The stationarity implies that if Lagrangian is independent of u_i, $\frac{\partial F}{\partial z}$ is independent of x_i, as it is evident from (**??**). If Lagrangian is independent of vector u, the first integral exists, and the Euler equations are

$$\frac{\partial F}{\partial z}=\text{ constant}, \quad z=\nabla\cdot u$$

If the boundary part of Lagrangian is zero, $F_\partial=0$, the minimizer

Problem 1.1 Derive Euler equations formally using the formula (**??**) for matrix differentiation and recalling that $\nabla\cdot u=\operatorname{Tr}(\nabla u)$.

Example 1.1 (Quadratic Lagrangian) Consider the problem

$$\min u\frac{1}{2}\int_\Omega\left((\nabla\cdot u)^2+\alpha^2u^2\right)dx+\frac{1}{2}\int_{\partial\Omega}(\beta u^2+2\gamma u)ds$$

where u is a two-dimensional vector minimizer, and $\Omega \subset R^2$. Lagrangian has the form

$$L = \frac{1}{2}(\nabla \cdot u)^2 + \frac{\alpha}{2}u^2 \quad F_\partial = \frac{\beta}{2}u^2 + \gamma u$$

The Euler equation is computed to be

$$-\nabla(\nabla \cdot u) + \alpha^2 u = 0$$

or, in coordinates,

$$\begin{aligned} \left(\alpha^2 + \frac{\partial^2}{\partial x_1^2}\right) u_1 + \frac{\partial^2}{\partial x_1 \partial x_2} u_2 &= 0, \\ \left(\alpha^2 + \frac{\partial^2}{\partial x_2^2}\right) u_2 + \frac{\partial^2}{\partial x_1 \partial x_2} u_1 &= 0 \end{aligned}$$

The natural boundary conditions (**??**) have the form

$$\begin{aligned} \nabla \cdot u + \beta(u \cdot n) + \gamma &= 0 \\ \beta(u \cdot t) + \gamma &= 0 \end{aligned}$$

Notice that tangent components $u \cdot n$ of the optimal boundary values of u are independent of the bulk part of Lagrangian F and are completely defined by the boundary part F_∂. The normal component $u \cdot n$ of the minimizer links the bulk and boundary parts.

Example 1.2 (Nonunique minimizer) Consider again the problem from the previous example, and assume that

$$\alpha = \beta = \gamma = 0$$

or $F = \frac{1}{2}(\nabla \cdot u)^2$, $F_\partial = 0$. Let us analyze the necessary conditions.

The Euler equation becomes $\nabla(\nabla \cdot u) = 0$. It allows for the first integral $\nabla \cdot u = c = \text{ constant}(x)$ in Ω. The boundary condition $(\nabla \cdot u)n = 0$ becomes $c\boldsymbol{n} = 0$, it is satisfied when $c = 0$. Thus, a stationary solution satisfies condition

$$\nabla \cdot u = 0 \quad \text{ in } \Omega.$$

The minimizer u is clearly nonunique. To analyze the nonuniqueness, we use Helmholtz representation

$$u = \nabla \phi + \nabla \times A$$

The stationarity condition are satisfied if $\phi = 0$ and u is an arbitrary solenoidal vector $u = \nabla \times A$.

1.3 Lagrangian dependent of Curl

Consider the Lagrangian $F(x, u, \nabla\times u)$ where $F(x, y, z)$ is the twice-differentiable function of three d-dimensional vectors x, y, and z and the differentiable boundary Lagrangian $f(s, u)$. Consider a variational problem for a vector minimizer u,

$$\min_u J \quad J = \int_\Omega F(x, u, \nabla \times u)dx + \int_{\partial\Omega} F_\partial(s, u)ds.$$

We derive Euler equation is a standard manner, computing the linearized increment of I as

$$I(u + \delta u) - I(u) = \int_\Omega \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial(\nabla \times u)} \cdot \nabla \times (\delta u) \right) dx$$
$$+ \int_{\partial\Omega} \frac{\partial F_\partial(s, u)}{\partial u} ds + o(\|\delta u\|).$$

Integrating by parts of the last term under bulk integral in the right-hand side using Stokes' theorem (**??**),

$$\int_\Omega \frac{\partial F}{\partial(\nabla \times u)} \cdot \nabla\times(\delta u)dx = -\int_\Omega \nabla\times \frac{\partial F}{\partial(\nabla \times u)} \cdot \delta u dx + \int_{\partial\Omega} \frac{\partial F}{\partial(\nabla \times u)} \times n(\delta u)ds,$$

we arrive at the vector Euler-Lagrange equation in the form:

$$-\nabla \times \frac{\partial F}{\partial(\nabla \times u)} + \frac{\partial F}{\partial u} = 0 \quad \text{in } \Omega. \tag{5}$$

The variational boundary condition are

$$\frac{\partial F}{\partial(\nabla \times u)} \times n + \frac{\partial F_\partial}{\partial u} = 0.$$

If $F_\partial = 0$, the stationarity conditions correspond to underdetermined boundary value problem. Indeed, it consists of three second-order differential equations (5) in Ω and the boundary conditions (**??**) that determines two scalar conditions

$$\frac{\partial F}{\partial(\nabla \times u)} \cdot t_i = 0, \; i = 1, 2 \quad \frac{\partial F}{\partial(\nabla \times u)} \cdot n - \text{is arbitrary},$$

where t_1, t_2 are two orthogonal tangents to the boundary surface.

First integral Assume that Lagrangian is independent of u, $F = F(\nabla \times u)$. By Helmholtz theorem (4) the curlfree term $\frac{\partial F}{\partial(\nabla\times u)}$ is a gradient of a potential ϕ,

$$\frac{\partial F}{\partial(\nabla \times u)} = \nabla\phi$$

Example 1.3 (Maxwell Equations) Lagrangian for the Maxwell equation in vacuum is expressed through the scalar electric potential Z_0 and the vector magnetic potential $\boldsymbol{Z} = [Z_1, Z_2, Z_3]$. It has the form (see []).

$$\frac{1}{8\pi}\left\{\left(\nabla Z_0 - \frac{\partial \boldsymbol{Z}}{\partial t}\right)^2 - \left(\nabla \cdot \boldsymbol{Z} - \frac{\partial Z_0}{\partial t}\right)^2 - (\nabla \times \boldsymbol{Z})^2\right\} \tag{6}$$

Using the derived formulas, we obtain the stationarity conditions with respect to $\boldsymbol{Z}$ and Z_0 as

$$\delta \boldsymbol{Z}: \quad -\frac{\partial}{\partial t}\left(\nabla Z_0 - \frac{\partial \boldsymbol{Z}}{\partial t}\right) - \nabla \times \nabla \times \boldsymbol{Z} - \nabla\left(\nabla \cdot \boldsymbol{Z} - \frac{\partial Z_0}{\partial t}\right) = 0$$

and

$$\delta Z_0: \quad \nabla \cdot \left(\nabla Z_0 - \frac{\partial \boldsymbol{Z}}{\partial t}\right) + \frac{\partial}{\partial t}\left(\nabla \cdot \boldsymbol{Z} - \frac{\partial Z_0}{\partial t}\right) = 0$$

respectively.

After the simplification, and the use of (**??**) they take the canonic form of the Maxwell equations:

$$-\frac{\partial^2}{\partial t^2}\boldsymbol{Z} + \nabla^2 \boldsymbol{Z} = 0, \quad -\frac{\partial^2}{\partial t^2} Z_0 + \nabla^2 Z_0 = 0 \tag{7}$$