Design of sources

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1 Pointwise constraints: Optimal design

1.1 Stationarity conditions

Consider an optimal design problem. The statement of the problem requires the definition of the goal functional, differential constraints (the equations of the equilibrium of dynamics), the possible integral constraints, and the set $\mathcal{K}$ of controls. It may look as follows: Minimize a functional of the type

$$\Phi = \min_{k(x)\in\mathcal{K}} \left[\int_\Omega F(k,u)\,dx + \int_{\partial\Omega} F_\partial(k,u)\,ds \right] \tag{1}$$

where $u = u(x_1, \ldots, x_d)$ is a solution of a partial differential equation

$$Q(k, \nabla u, u) = 0 \text{ in } \Omega, \quad u = u_0 \text{ on } \partial\Omega \tag{2}$$

The approach with pointwise algebraic or differential constraints is similar those used in to one-dimensional variational problems. We construct an augmented functional I, adding to the functional the differential equation multiplied by a Lagrange multiplier $\lambda(x)$- the function of x, as follows

$$I = \min_{k(x)\in\mathcal{K}} \left[\min_{u:\ u=u_0 \text{ on } \partial\Omega} \left(\max_\lambda J(u,k,\lambda) \right) \right] \tag{3}$$

$$J = \int_\Omega \left[F(k,u) + \lambda\, Q(\nabla u, u) \right] dx + \int_{\partial\Omega} F_\partial(k, u_0)\,ds \tag{4}$$

We arrive at the variational problem with three variables, $u, k,$ and λ. The stationarity condition with respect of μ is equation (5). The other two conditions are

$$S_u(F + \lambda Q) = 0, \quad S_\lambda(F + \lambda Q) = 0, \text{ in } \Omega \tag{5}$$

and corresponding boundary conditions.

Below we consider two simple minimization problems with differential constraints that express a thermal equilibrium. The equilibrium depends on the control (thermal sources or boundary temperature) that must be chosen to minimize a functional related to the temperature distribution. The differential constraint is the conductivity equation; it relates the temperature and control.

1.2 Design of boundary temperature

Consider the following problem: A bounded domain Ω is in thermal equilibrium. The temperature on its boundary $\theta(s)$ must be chosen to minimize the L_2 norm of deflection of the temperature T from a given target distribution $\rho(x)$.

Let us formulate the problem. The objective is

$$I = \min_T \frac{1}{2} \int_\Omega (T - \rho)^2 dx. \tag{6}$$

Temperature T is a solution to the boundary value problem of thermal equilibrium (the differential constraint)

$$\nabla^2 T = 0 \quad \text{in } \Omega, \quad T = \theta \quad \text{on } \partial\Omega. \tag{7}$$

Here, control θ enters the problem through the boundary condition of the differential constraint. The constraint (7) connects control $\theta(s)$ with the variable $T(x)$, which is needed to compute the objective. The set of controls θ is an open set of all piece-wise differentiable functions.

Harmonic target The problem becomes trivial when target ρ is harmonic, $\nabla^2 \rho = 0$ in Ω. In this case, we simply set $T = \rho$ everywhere in Ω and in particular at the boundary. The differential constraint is thus satisfied. The cost of the problem is zero, which mean that the global minimum is achieved.

Nonharmonic target: Stationarity We account for the first equation (7) as for the pointwise constraint. The Lagrange multiplier for the differential constraint (called also the adjoint variable) $\lambda(x)$ is a function of a point of the domain, because the constraint is enforced everywhere there. The augmented functional is

$$I_A = \int_\Omega \left(\frac{1}{2}(T - \rho)^2 + \lambda \nabla^2 T \right) dx. \tag{8}$$

It reaches stationarity at the optimal solution. To compute variation of I_A with respect to T, we twice integrate by parts the last term. The computation gives the following

$$\delta I_A = \int_\Omega \left(T - \rho + \nabla^2 \Lambda\right) \delta T \, dx + \oint_{\partial\Omega} \left[\left(\delta \frac{\partial T}{\partial n} \right) \lambda - \delta T \left(\frac{\partial}{\partial n} \lambda \right) \right] ds \tag{9}$$

The variation leads to the stationarity condition, which here has the form of the boundary value problem for λ,

$$\nabla^2 \lambda + T = \rho \quad \text{in } \Omega, \tag{10}$$

and

$$\lambda = 0, \quad \frac{\partial}{\partial n} \lambda = 0 \quad \text{on } \partial\Omega \tag{11}$$

Notice that the variations (see (7)) of the value of T and its normal derivative $\frac{\partial T}{\partial n}$ at the boundary $\partial\Omega$ are arbitrary because the control θ is not constrained, therefore the coefficients by these variations must be zero at the stationary solution.

Remark 1.1 The problem (10) for the dual variable λ has two boundary conditions, and the problem (7) for T has none! However, these two problems form a system of two second-order partial differential equations with two boundary conditions (11). The primary problem is underdetermined because the control cannot be specified without the dual problem. The dual problem is overdetermined because it determines the control. The system is well-posed.

Analysis To solve the system of necessary conditions, we first exclude T by taking Laplacian ∇^2 of the left- and right-hand side of the equation (10) and accounting for (7). Thus, we obtain a regular fourth-order boundary value problem for λ

$$\nabla^4 \lambda = \nabla^2 \rho \quad \text{in } \Omega, \quad \lambda = 0, \quad \frac{\partial}{\partial n}\lambda = 0 \quad \text{on } \partial\Omega. \tag{12}$$

that has a unique solution. After finding λ, we find T from (10). Then we compute the boundary values $\theta = T|_{\partial\Omega}$ and define the control.

If the target is harmonic, $\nabla^2 \rho = 0$, then (12) gives $\lambda = 0$ and (10) gives $T = \rho$, as expected.

1.3 Design of bulk sources

Again, consider problem (6) of the best approximation of the target temperature. This time, consider control of bulk sources. Namely, assume that the heat sources $\mu = \mu(x)$ can be applied everywhere in the domain Ω, but the boundary temperature is kept equal zero. Assume also, that the L_2 norm of the sources is bounded.

In this case, T is a solution to the boundary value problem (the differential constraint)

$$\nabla^2 T = \mu \quad \text{in } \Omega, \quad T = 0 \quad \text{on } \partial\Omega \tag{13}$$

and μ is bounded by an integral constraint

$$\frac{1}{2}\int_\Omega \mu^2 dx = A \tag{14}$$

but not pointwise. These constraints are accounted with Lagrange function $\lambda(x)$ and Lagrange multiplier γ, respectively. The extended functional depends on two functions T and μ,

$$\int_\Omega L(T, \mu) dx - \gamma A$$

where

$$L(T, \mu) = \frac{1}{2}(T - \theta)^2 + \lambda(\nabla^2 T - \mu) + \frac{1}{2}\gamma \mu^2 \tag{15}$$

The variations of L with respect to T and μ lead to stationary conditions. The stationarity with respect to T results in the boundary value problem for λ,

$$\nabla^2 \lambda = T - \theta \quad \text{in } \Omega, \quad \lambda = 0 \quad \text{on } \partial\Omega \tag{16}$$

The stationarity with respect to variation of μ leads to the point-wise condition

$$\lambda = -\gamma \mu$$

that allows to exclude μ from (13) and obtain the linear system (Recall, that γ is a constant)

$$\begin{aligned} \nabla^2 T &= -\frac{1}{\gamma}\lambda \quad \text{in } \Omega, \quad T = 0 \quad \text{on } \partial\Omega \\ \nabla^2 \lambda &= T - \theta \quad \text{in } \Omega, \quad \lambda = 0 \quad \text{on } \partial\Omega \end{aligned}$$

and an integral constraint

$$\frac{1}{2}\int_{\Omega}\lambda^2 dx = \frac{A}{\gamma^2}.$$

This system could be solved for $T(x)$, $\lambda(x)$ and the constant γ, which would completely define the solution.

Problem 1.1 Reduce the system to one fourth-order equation as in the previous problem. Derive boundary conditions. Using Green's function, obtain the integral representation of the solution through the target.