Final Exam
M 3150-001, Spring 2017

Name ....

Solve the following problems. Show the method (the derivation),

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1. Find the Fourier series of

$$f(x) = \begin{cases} 0 & \text{if } x \in (-1, 0) \\ 1 & \text{if } x \in (0, 1) \end{cases} \quad , \quad x \in (-1, 1).$$

   Evaluate the first four coefficients of the series.

2. A 1-periodic function $g(t)$, $t \in (0,1)$ is presented by its Fourier series

$$g(t) = \sum_{n=1}^{\infty} a_n \sin(\pi\, n\; t)$$

with some unknown coefficients a_n. This function satisfies the differential equation

$$g''(t) + c^2\, g(t) = \sum_{n=1}^{\infty} \frac{1}{\pi^3 n^3} \sin(\pi\, n\, t)$$

Find the coefficients a_n.

3. Find $u(x,t)$, that satisfies the following conditions:

Heat Equation	$\dfrac{\partial u}{\partial t} = 2\,\dfrac{\partial^2 u}{\partial x^2}, \quad x \in [0,1],\ t \in [0,\infty).$
Boundary cond.	$u(0,t) = 0, \quad u(1,t) = 0.$
Initial cond.	$u(x,0) = 2\sin(7\pi x)\,.$

4. Determine the equilibrium temperature distribution $u(r, \theta)$ *outside* of a disk of radius $a = 1$, if the boundary temperature is $u(1, \theta) = 2 + \cos(5\theta)$, and the temperature at infinity is bounded. Show the separation of variables technique.

   Note: Laplacian in polar coordinates has the form:

$$\nabla u(r, \theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

5. A string vibrates according to the equation

$$\frac{\partial^2 u}{\partial t^2} = 9\,\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the deflection, $x \in [0,1]$, $t \in [0,\infty)$.
The ends are fixed: $u(0,t) = 0$, $u(1,t) = 0$,
the initial shape $u(x,0) = f(x)$ is $f(x) = \sin(\pi x) - 2\sin(5\pi x)$, and
the initial speed is zero, $\frac{\partial u}{\partial t}|_{t=0} = 0$.
Find $u(x,t)$.

6. The initial distribution of the temperature in an infinite bar is

$$u(x, 0) = e^{-4x^2}$$

The diffusivity constant k is equal two, $k = 2$. Find the temperature distribution $u(x, t)$ at any time t using convolution formula.

Hint: The inverse Fourier transform $\mathcal{F}^{-1}$ of $G(\omega) = e^{-k\omega^2 t}$ is

$$\mathcal{F}^{-1}(G(\omega)) = \sqrt{\frac{\pi}{kt}} e^{-\frac{x^2}{4kt}}$$