To the Final: Problems for practice. M 3150-001, Spring 2017

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Solve the following problems. Show the method (the derivation), do not simply copy the final formulas.

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1. Find $u(x,t)$, that satisfies the following conditions:

$$\text{Heat Equation} \quad \frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad x \in [0,2], \; t \in [0,\infty).$$

$$\text{Boundary cond.} \quad u(0,t) = 0, \quad u(2,t) = 0.$$

$$\text{Initial cond.} \quad u(x,0) = 3\sin(\pi x).$$

2. Determine the equilibrium temperature distribution $u(r,\theta)$ inside a disk of radius a $= 1$, if the boundary temperature is $u(1,\theta) = 3\cos(3\theta)$. Note: Laplacian in polar coordinates has the form

$$\nabla u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

3. Find the sine Fourier series of

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \in [1,3] \end{cases}, \quad x \in (0,3).$$

Find the first three nonzero coefficients of the series.

4. The sine Fourier series of $f(x) = x\,(1-x)$, $x \in (0,\,1)$ is

$$f(x) = \sum_{n=1}^{\infty} \sin(\pi\, n\, x), \quad a_n = \frac{2}{\pi^3 n^3}\left((-1)^n - 1\right)$$

Sketch the graphs of $f(x)$, $f'(x)$, $f''(x)$, $x \in (0,\,1)$ and find Fourier series of them. Find the first two nonzero coefficients of the series.

5. A string vibrates according to the equation

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the deflection, $x \in [0,1]$, $t \in [0,\infty)$.
The ends are fixed: $u(0,t) = 0$, $u(1,t) = 0$,
the initial shape $u(x,t)$ is $u(x,0) = 2\sin(5\pi x)$, and
the initial speed is zero, $\frac{\partial u}{\partial t}|_{t=0} = 0$.
Find $u(x,t)$.

6. The initial distribution of the temperature in an infinite bar is

$$u(x,0) = \begin{cases} 1 & \text{if } x \in [-1,1] \\ 0 & \text{if } x \notin [-1,1] \end{cases}$$

Find the temperature distribution $u(x,t)$ at any time t. The diffusivity constant is one.