Second Variation II

February 20, 2013

Contents

1 Weierstrass-type Test

1.1 Localized variations

Convexity of Lagrangians and Stability of Solutions We have shown in Chapter **??** that a solution to a one-dimensional variational problem is stable against fine-scale perturbations if its Lagrangian is convex. The lack of convexity of the Lagrangian leads to the appearance of rapidly alternating functions in the optimal solution. Here we develop a similar approach for multidimensional variational problems.

Weierstrass-Type Variation The Weierstrass-type condition checks that a sharp localized perturbation of the extremal does not decrease the functional. Failure to satisfy the Weierstrass test proves that the checked stationary solution is not optimal because it can be improved by adding an oscillatory component to it. We define the local strong perturbation (or the strong local variation or Weierstrass variation) as follows.

Definition 1.1 By a strong local variation (Weierstrass-type variation) of a multidimensional variational problem we understand a localized perturbation δu of the potential u that

1. is differentiable almost everywhere,
2. has an arbitrarily small magnitude $|\delta u| < \varepsilon$;
3. has a finite magnitude of the gradient $|\nabla u| = O(1)$; and
4. is localized in a small neighborhood $\omega_\varepsilon(x_0)$ of an inner point x_0 in the domain Ω: $\delta u(x) = 0 \; \forall \; x \notin \omega_\varepsilon(x_0)$, where $\omega_\varepsilon(x_0)$ is a domain in Ω with the following properties: $x_0 \in \omega_\varepsilon(x_0)$, $\text{diam}(\omega_\varepsilon) \to 0$ as $\varepsilon \to 0$.

There is a freedom in choosing the type of Weierstrass variation in a multidimensional problem. First, we choose the shape of ω_ε. It is important that δw is continuous and vanishes on the boundary $\partial\omega_\varepsilon$ of ω_ε. For example, we may choose ω_ε as a circular domain and consider the trial perturbation δw shaped like a cone, or a symmetric paraboloid. For a polygonal domain ω_ε the variation δw can be shaped like a pyramid.

The increment of the functional due to this variation is of the order of the size of the domain of variation ω_ε. The main term of the increment depends on the perturbation ∇u is the Lagrangian is Lipschitz with respect to u and x and coercive with respect to ∇u (which we will assume from now),

$$F(x, u + \delta u, \nabla(u + \delta u)) = F(x_0, u, \nabla u + \delta v)) + o(\varepsilon), \quad u = u(x_0), \quad \forall x \in \omega_\varepsilon$$

where x_0 is the coordinate of the center of ω_ε,

$$\delta I_W(\omega_\varepsilon) = \int_{\omega_\varepsilon} [F(x_0, u, \nabla u + \nabla v) - F(x_0, u, \nabla u)]\, dx + o\|\omega_\varepsilon\|, \qquad (1)$$

Notice that the only variable in ω_ε argument is ∇v, but the slow variable u is "frozen" to be equal to be the values of the checked stationary solution. The independent variable x is replaced by x_0 without change in the main term of the increment. The integral of ∇v over ω_ε is zero,

$$\int_{\omega_\varepsilon} \nabla v \, dx = V \otimes \int_{\omega_\varepsilon} \nabla s \, dx = 0 \tag{2}$$

The resulting necessary condition depends on the chosen shape of the variation. We will call the corresponding inequalities the necessary conditions of Weierstrass type or the *Weierstrass conditions*. The Weierstrass condition depends on magnitude of the variation as well as on the shape of the domain ω_ε.

1.2 Conical variation

Let $u = (u_1, \ldots, u_n)$ be a stationary solution to the problem (**??**) and $\Omega \subset R_2$. Consider the localized perturbation δu_i of the potential u_i that is shaped as a cone with the vertex at the origin. Assume that the cone is supported by a closed polar curve $\rho = \varepsilon b(\phi)$,

$$\omega_\varepsilon = \{(\rho, \phi) : \quad 0 \le \rho \le \varepsilon b(\phi), \quad -\pi < \phi \le \pi\}$$

where $b(\phi) > 0$ for all ϕ, and the hight of the cone is εV_i. The shape of the perturbation, in the cylindrical coordinates, is $\delta u_i(\rho, \phi) = V_i s(\rho, \phi)$ where

$$s(\rho, \phi) = \begin{cases} \varepsilon \left(1 - \frac{\rho}{\varepsilon b(\phi)}\right) & \text{if } \rho \le \varepsilon b(\phi) \\ 0 & \text{if } \rho > \varepsilon b(\phi) \end{cases}$$

is a cone. Each potential is varied alike. The variation of the vector potential is $\delta v(\rho, \phi) = V s(\rho, \phi)$ where $V = (V_1, \ldots, V_n)$ is the vector of magnitudes of the variations of the potentials $u_1, \ldots, u_n$. The gradient ∇v of the variation is a dyad of the form

$$\nabla v(\phi) = V \otimes P \tag{3}$$

where

$$P(\rho, \phi) = \nabla s(\phi) = \begin{cases} \left(-\frac{1}{b(\phi)} i_\rho + \frac{b'(\phi)}{b^2(\phi)} i_\phi\right) & \text{if } \rho \in \omega_\varepsilon \\ 0 & \text{if } \rho \notin \omega_\varepsilon \end{cases} \tag{4}$$

The gradient of the variation is of the order $o(1)$, the variation itself is of the order of s that is of the order of ε, and the diameter of the domain ε is also of the order ε. The variation fits Definition 1.1 of the Weierstrass type-variation.

The arguments u, x are equal to be the values of the checked stationary solution. The variable argument is $\nabla v(\rho, \phi)$ of the form (4) is independent on ρ inside $\omega\varepsilon$,

$$\nabla v(\rho, \phi) = V \otimes P(\phi) \quad \text{ in } \omega_\varepsilon$$

Its rank is equal to one in each point. Moreover, the integral of ∇v over any subdomain $\omega \subset \Omega$ is rank-one too,

$$\int_\omega \nabla v\, dx = V \otimes \int_\omega \nabla s\, dx$$

because vector V in (4) is independent of the coordinates.

To compute the increment, it remains to rewrite integral in the polar coordinates as

$$\delta I_W(\omega_\varepsilon) = \int_{-\pi}^{\pi} \left(\int_0^{\varepsilon b(\phi)} \left[F(u, \nabla u + V \otimes P(\phi)) - F(u, \nabla u) \right] \rho d\rho \right) d\phi \qquad (5)$$

subject to the constraint (2).

The inner integral over a infinitesimal sector is immediately evaluated because no argument depends on ρ (the direction of the gradient P is constant in a sector, its value is constant in $[0, b]$), therefore

$$\delta I_W(\omega_\varepsilon) = \frac{1}{2} \int_{-\pi}^{\pi} b^2(\phi) \left(F(u, \nabla u + V \otimes P(\phi)) - F(u, \nabla u) \right) d\phi \qquad (6)$$

The constraint (2) takes the form

$$\int_{-\pi}^{\pi} b^2(\phi) P(\phi)) d\phi = 0 \qquad (7)$$

Next, the shape $b(\phi)$ of the variation domain ω_ε must be specified. The strength of the obtained condition depends on the shape of the domain, therefore one would try to choose the "most dangerous domain of variation by minimizing the normalized increment.

Summing up, we formulate the necessary condition

Theorem 1.1 (Stability to Weierstrass-type variation in a cone) Every stationary solution that corresponds to minimum of the functional (**??**) satisfies the inequality $I_w \geq 0$, where

$$B_W = \min_{\omega_\varepsilon} \delta I_W(\omega_\varepsilon) \quad \|\omega_\varepsilon\| = \varepsilon \qquad (8)$$

Otherwise the stationary solution u can be improved by adding a conical perturbation to the solution.

The Weierstrass condition in a conical domain corresponds to stability of the solution to a specified type of perturbation. The violation of the condition shows the way to improve solution; in this sense the condition is constructive.

1.3 Rank-One Convexity

Variation in a symmetric cone or strip The variation (5) can be simplified when ad additional assumption of the symmetry of ω_ε is made. Assume that its boundary $b(\phi)$ is symmetric to the rotation on 180°, $b(\phi) = b(\phi + 180°)$. Then $P(\phi) = -P(\phi + 180°)$. The variation (5) becomes

$$\delta I_W(\omega_\varepsilon) = \frac{1}{2}\int_0^\pi D_I(\phi) b^2(\phi) d\phi \tag{9}$$

where

$$D_I(\phi) = F(u, \nabla u + V \otimes P(\phi)) + F(u, \nabla u - V \otimes P(\phi)) - 2F(u, \nabla u) \tag{10}$$

The necessary condition (8) is satisfied is

$$D_I(\phi) \geq 0 \quad \forall \phi \in [0, Pi). \tag{11}$$

Notice that the last condition is equivalent to the convexity of $F(u, \nabla u)$ in the "direction" $V \otimes P$. This "direction" is an arbitrary $n \times 2$ dyad because both vectors V and P are arbitrary.

Variation in the parallel strips The opposite type of variation corresponds to the extremely elongated rectangular domain ω_ε of the size $(\varepsilon, \varepsilon^2)$ that consists of several thin strips parallel to the longer side. The variation of the potential depends on the normal n to the strips everywhere except in the end domains ϵ_{ε^2} of the size $\varepsilon^2 \times \varepsilon^2$ at the ends of the strips where it monotonically decays to zero. Assume that the potential is piece-wise linear continuous function of n. Its magnitude is of the order ε^2 of the thickness c_k of the layers. The gradient ∇v, however, is a piece-wise constant vector function with the values V_k of the finite magnitude everywhere except of the end domains ϵ_{ε^2} where if is bounded. The contribution of these domains is arbitrary small comparing with the contribution of the much larger middle section $\epsilon_\varepsilon - \epsilon_{\varepsilon^2}$ of the domain. The main term in the increment comes from the variation the middle part of the strip. Here, the gradient $\nabla v_i = \nu_i(n) n$ of each potential v_i is directed along the normal n to the strips. Function ν is piece-wise constant and takes a constant value V_{ki} in each strip. The variation of the vector potential $v = \{v_1, \ldots, v_n\}$ has the form

$$\nabla v(n) = V(n) \otimes n, \qquad \text{in } \epsilon_\varepsilon - \epsilon_{\varepsilon^2}$$

where $V(n)$ is a piece-wise constant function with the values $V_k = (v_{k1}, \ldots, v_{kn}$ in the kth strip.

The perturbation of the potential v is zero outside of the ω_ε and is continuous inside the domain; this leads to the constraint on the magnitudes V_k

$$\sum_k c_k V_k = 0, \quad \sum_k c_k = 1, \quad c_k \geq 0 \tag{12}$$

here c_k is the relative thickness of the kth strip.

The increment ΔI of the cost of the variational problem (**??**) due to the variation in the strip is

$$\Delta I = \sum_k c_k F(u, \nabla u + V_k \otimes n) - F(u, \nabla u) \tag{13}$$

Solution u is stable to the perturbation in a strip if

$$\Delta I > 0 \quad \forall, V_k, c_k \text{ as in (12) }, \quad \forall n$$

Rank-One convexity The condition (**??**) states that the Lagrangian $\alpha F(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{A})$ is convex with respect to some special trial matrices of the type $\boldsymbol{R} = \boldsymbol{\alpha} \otimes \boldsymbol{n}$ but not with respect to arbitrary matrices. The corresponding property is called the rank-one convexity.

Definition 1.2 The scalar function F of an $n \times m$ matrix argument $\boldsymbol{A}$ is called *rank-one convex* at a point $\boldsymbol{A}_0$ if

$$F(\boldsymbol{A}_0) \leq \sum_{i=1}^N \alpha_i F\left(\boldsymbol{A}_0 + \alpha_i \xi_i \boldsymbol{R}\right) \tag{14}$$

for any $\alpha_i,\ \xi_i,\ \boldsymbol{R},\ N$ that

$$\sum_{i=1}^N \alpha_i = 1, \quad \alpha_i \geq 0, \quad \sum_{i=1}^N \alpha_i \xi_i = 0, \quad \boldsymbol{R} = \boldsymbol{a} \otimes \boldsymbol{b}.$$

Here $\boldsymbol{a}$ and $\boldsymbol{b}$ are n-dimensional and m-dimensional vectors, respectively, and α_i are scalars.

Rank-one convexity requires convexity in some matrix "directions," namely, in the "directions" of the rank-one matrices. Obviously, the usual convexity implies rank-one convexity.

There are two cases in which rank-one convexity coincides with convexity:

1. The Lagrangian depends on one independent variable: $\boldsymbol{x}$ is a scalar.

2. The Lagrangian depends on one dependent variable: $\boldsymbol{w}$ is a scalar.

In both cases, the matrix $\boldsymbol{A}_0 = \nabla \boldsymbol{w}$ degenerates into a rank-one matrix. These cases are studied in previous chapters: Chapter 1 deals with one-dimensional problems, and Chapters 4 and 5 discuss the scalar problem. The corresponding variational problems are discussed in [**?**].

Example 1.1 (Non-convex but rank-one convex function) Let A be a 2×2 matrix and $F(A)$ be

$$F(A) = [\ \mathrm{Tr}\,(A)]^2 + 2C \det A \tag{15}$$

We show that $F(A)$ is nonconvex, if $C \neq 0$, but it is rank-one convex for all real C. Indeed, F is a quadratic form, $F(A) = A_v^T M \, A_v$, of the elements of A that form the four-dimensional vector $A_v = (a_{11}, a_{22}, a_{12}, a_{21})$. Matrix M of this form is

$$M = \begin{pmatrix} 1 & 1+C & 0 & 0 \\ 1+C & 1 & 0 & 0 \\ 0 & 0 & 0 & -C \\ 0 & 0 & -C & 0 \end{pmatrix}$$

Its eigenvalues are $C, C+2, \pm C$. At least one of the eigenvalues is negative if $C \neq 0$, which proves that $F(A)$ is not convex.

Compute the rank-one perturbation of F. We check by the direct calculation that

$$\sum_k c_k \det\left(A + \alpha_k d \otimes b\right) = \det A,$$

if

$$\sum_k c_k \alpha_k = 0 \tag{16}$$

Indeed, all quadratic in the elements of $d \otimes b$ terms in the left-hand side cancel, and the linear terms sum to zero because (16). We also have

$$\left(\sum_k c_k \ \mathrm{Tr}\, F(A + \alpha_k d \otimes b)\right)^2 = (\ \mathrm{Tr}\, A)^2 + \left(\sum_k c_k \alpha_k \ \mathrm{Tr}\,(d \otimes b)\right)^2$$

(linear in $d \otimes b$ terms cancel because of (16)).

Substituting these two equalities into F in (15), we find that

$$\sum_k c_k F(A + \alpha_k d \otimes b) = F(A) + \left(\sum_k c_k \ \mathrm{Tr}\,(\alpha_k d \otimes b)\right)^2$$

if if (16) holds. The variation is independent of the value of C. The inequality (14) follows; therefore F is rank-one convex.

Stability of the stationary solution The rank-one convexity of the Lagrangian is a necessary condition for the stability of the minimizer. If this condition is violated on a tested solution, then the special fine-scale perturbations (like the one described earlier) improve the cost; hence the classical solution is not optimal.

Theorem 1.2 (Stability to Weierstrass-type variation in a strip) Every stationary solution that corresponds to minimum of the functional (**??**) corresponds to rank-one convex Lagrangian. Otherwise the stationary solution u can be improved by adding a perturbation in a strip to the solution.

Remark 1.1 Rank-one trial perturbation is consistent with the classical form $L(\boldsymbol{x}, \boldsymbol{w}, \nabla \boldsymbol{w})$ of Lagrangian. This form implies the special differential constraints $\nabla \times (\boldsymbol{v}) = 0$ that require the continuity of all but one component of the field $\nabla \boldsymbol{w}$. The definition of this necessary condition for the stability of the solution can be obviously generalized to the case where the differential constraints are given by the tensor $\mathcal{A}$.

1.4 Legendre-type condition

A particular case of the Weierstrass-type condition is especially easy to check. If we assume in addition that the magnitude V of the variation is infinitesimal, the rank-one condition becomes the requirement of positivity of the second derivative in a rank-one "direction"