HW 4 Calc. Variation 2013

Due in two weeks, at March 8

1. Check the Weierstrass condition and find a minimizing sequence that brings the goal to zero
$$\inf_{u(0)=u(1)=0} \int_0^1 F(u,u')dx, \quad F = \min\left\{(u'-1)^2, (u'+1)^2\right\} + u^2$$
What is the minimizing sequence if the boundary conditions are
(a) $u(0) = 10, u(1) = 1$
(b) $u(0) = .5, u(1) = 0.$

2. Using cylindrical coordinates (r, θ, z), find the geodesic (shortest path) on the cylinder $r = 1$ that joints the points $(1, 0, 0)$ and $(1, \pi, 1)$. Show that a family of helixes corresponds to stationary solution. Select the true geodesics by using Jacobi condition.

3. Derive the convex envelope of the following functions
$$f_1(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = 0,\ x_2 = 0 \\ 1 + x_1^2 + x_2^2 & \text{if } x_1^2 + x_2^2 \neq 0 \end{cases}$$
and
$$f_2(x_1, x_2) = \min\Big\{(x_1+1)^2 + (x_2+1)^2,\ (x_1-1)^2 + (x_2+1)^2,$$
$$(x_1+1)^2 + (x_2-1)^2,\ (x_1-1)^2 + (x_2-1)^2\Big\}$$
(In the last case, the envelope is a multiface surface.) Qualitatively describe how the convex envelope in the proximity of $(0, 0)$ changes if the component $(x_1+1)^2+(x_2+1)^2$ is replaced by $(x_1+1)^2+(x_2+1)^2+\epsilon$, where $|\epsilon| \ll 1$.

   Graph the solutions using Maple or similar package.

4. Using Gradient scheme, compute and plot the first five iterations of approximate solution for brachistrochrone problem. The boundary conditions are: $u(0) = 1,\ u(1) = 0$. Use a parabola as a first approximation. Compare with the exact solution.