HW 3

M 5500 Calculus of Variations. Spring 2013

February 4, 2013

1. Derive the Euler-Lagrange equation for the Lagrangian
$$F = F(x, u, \nabla \times u),$$
where $x = (x_1, x_2, x_3)$ is a point in R_3 and $u = (u_1, u_2, u_3)$ is a vector minimizer. Use Stokes theorem for integration by parts. Work on the example
$$F = (\nabla \times u)^T (\nabla \times u) - c^2 u^2,$$

2. Derive the Euler-Lagrange equation for the Lagrangian
$$F = F(x, u, \nabla \cdot u),$$
where $x = (x_1, x_2, x_3)$ is a point in R_3 and $u = (u_1, u_2, u_3)$ is a vector minimizer.

3. Show that $\det(\nabla u(x))$ is a Null-Lagrangian, if $x = (x_1, x_2, x_3)$ is a point in R_3 and $u = (u_1, u_2, u_3)$.

4. Elastic energy of the bending plate is given by the formula
$$E = \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}\right)^2 - 2(1-\nu)\left(\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} - \left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2\right)\right]$$
Derive Euler-Lagrange equation and natural boundary conditions (conditions at the free edge). Discuss the dependence on ν and identify null-Lagrangian in the energy.