Hamiltonian

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Contents

1 Variational problem as a constrained problem

1.1 Differential constaint

The simplest variational problem

$$I = \min_u \int_0^1 F(x, u, u')dx$$

can be rewritten as the minimization of the constrained problem

$$I = \min_{u,v} \int_0^1 F(x, u, v)dx \quad \text{subject to } u' = v \tag{1}$$

where the constrain specifies the differential dependence between arguments of the Lagrangian. The last problem is naturally rewritten using Lagrange multiplier p:

$$I = \min_{u,v} \max_p \int_0^1 \left[F(x, u, v) + p(u' - v)\right] dx \tag{2}$$

Let us analyze the last problem. Interchange the sequence of extremal operations, obtaining the inequality:

$$I \geq I^D; \quad I^D = \max_p \int_0^1 \left(\min_{u,v} \int_0^1 [F(x,u,v) - p'u - p\,v]\, dx \right) dx + p\,u|_0^1 \quad (3)$$

(Here, we integrated by parts the term pu'). Write down the stationarity conditions for the minimal problem in brackets; notice that this problem include u and v but not their derivatives, therefore the optimization is performed independently in the each point of the trajectory:

$$I^D = \max_p \int_0^1 \int_0^1 F_D(x,p,p')dx + p\,u|_0^1 \quad (4)$$

where

$$F_D(x,p,p') = \min_{u,v} [F(x,u,v) - p'u - p\,v] \quad (5)$$

The coefficients by variations δu and δv are

$$p = \frac{\partial F}{\partial v}; \quad p' = \frac{\partial F}{\partial u} \quad (6)$$

respectively. Now, we may transform the problem in three different ways

1. Excluding p and p' from (6), we obtain the Euler equation:

$$\frac{d}{dx}\frac{\partial F}{\partial v} - \frac{\partial F}{\partial u} = 0, \quad u' = v$$

2. Excluding v from (6): $v = \rho(u,p)$ we express the problem through the Hamiltonian $H(u,p)$

$$I^H = \min_u \max_p \int_0^1 [u'p - H(u,p)]\, dx \quad (7)$$

where

$$H(u,p) = \rho(u,p)p - F(x,u,\rho(u,p)) \quad (8)$$

3. Excluding u and v from (6): $u = \phi(p,p')$, $v = \psi(p,p')$, we obtain the dual variational problem

$$I \geq I^D; \quad I^D = \max_p \int_0^1 \left[F^D(x,p,p')\right] dx \quad (9)$$

where

$$F^D(p,p') = F(x,\phi,\psi) - p'\phi - p\psi \quad (10)$$

1.2 Canonic form

Here, we derive Hamiltonian by rewriting the Euler equation in the canonic form.

The structure of Euler equations (**??**)

$$\frac{d}{dx}\frac{\partial L}{\partial u_i'} - \frac{\partial L}{\partial u_i} = 0, \quad i = 1, \ldots, N \tag{11}$$

can be simplified and unified if we rewrite them as $2N$ first-order differential equations instead of N second-order ones. A first-order system can be obtained from (11) if the new variables p_i are introduced,

$$p_i(x) = \frac{\partial L(x, u, u')}{\partial u_i'}, \quad i = 1, \ldots, N \tag{12}$$

In mechanics, $p = (p_1, \ldots, p_N)$ is called the vector of *impulse*. The Euler equation takes the form

$$p' = \frac{\partial L(x, u, u')}{\partial u} = f(x, u, u'), \tag{13}$$

where f is a function of x, u, u'. The system (12), (13) becomes symmetric with respect to p and u if we algebraically solve (12) for u' as follows:

$$u' = \phi(x, u, p), \tag{14}$$

and substitute this expression (13):

$$p' = f(x, u, \phi(x, u, p)) = \psi(x, u, p) \tag{15}$$

where ψ is a function of the variables u and p but not of their derivatives.

In summary, system (12), (13) is transferred to the canonic from (or Cauchy form)

$$\begin{array}{r} u' = \phi(x, u, p) \\ p' = \psi(x, u, p) \end{array} : \tag{16}$$

It is resolved for the derivatives u' and p' and is symmetric with respect to variables u and p. The properties of the solution are entirely determined by the algebraic vector functions ϕ, ψ in the right-hand side, which do not contain derivatives.

Remark 1.1 The equation (12) can be solved for u' and (14) can be obtained if the Lagrangian is convex function of u' of a superlinear growth. As we will see, (Chapter (**??**)), this condition is to be satisfied if the problem has a classical minimizer.

Example 1.1 (Quadratic Lagrangian) Assume that

$$L = \frac{1}{2}a(x)u'^2 + \frac{1}{2}b(x)u^2.$$

We introduce p as in (12)

$$p = \frac{\partial L(x,u,u')}{\partial u'} = au'$$

and obtain the canonic system

$$u' = \frac{1}{a(x)} p, \quad p' = b(x) u.$$

Notice that the coefficient $a(x)$ is moved into denominator.

The equations of Lagrangian mechanics (see below Section **??**) correspond to stationarity of the action

$$L(t,q,q') = \frac{1}{2}\dot{q}^T R(q)\dot{q} - V(q)$$

where R is the matrix of inertia, and $V(q)$ is a convex function called the potential energy. The impulses $p = \frac{\partial L}{\partial \dot{q}}$ are equal to $p = R(q)\dot{q}$. The canonic system becomes

$$\begin{aligned} &\dot{q} = R^{-1}p, \\ &\dot{p} = p^T \left(R^{-1}\tfrac{dR}{dq}R^{-1}\right) p - \tfrac{\partial V}{\partial q} \end{aligned}$$

The last equation is obtained by excluding $\dot{q} = R^{-1}p$ from the $\frac{\partial L}{\partial q} = \dot{q}^T \frac{dR}{dq}\dot{q} - \frac{\partial V}{\partial q}$.

1.3 Hamiltonian

We can rewrite the system (16) in a more symmetric form introducing a special function called *Hamiltonian*. The Hamiltonian is defined by the formula (see (16)):

$$H(x,u,p) = pu'(x,u,p) - L(x,u,u'(u,p)) = p\phi(x,u,p) - L(x,u,\phi(x,u,p)) \quad (17)$$

where u is a stationary trajectory – the solution of Euler equation. If the original variables u, u' are used instead of instead of u, p the expression for the Hamiltonian becomes

$$\hat{H} = u'\frac{\partial F}{\partial u'} - L(x,u,u') \quad (18)$$

To distinguish the function (18) from the conventional expression (**??**) for the Hamiltonian, we use the notation $\hat{H}$.

Let us compute the partial derivatives of H:

$$\frac{\partial H}{\partial u} = p\frac{\partial \phi}{\partial u} - \frac{\partial L}{\partial u} - \frac{\partial L}{\partial \phi}\frac{\partial \phi}{\partial u}$$

By the definition of p, $p = \frac{\partial L}{\partial u'} = \frac{\partial L}{\partial \phi}$, hence the first and third term in the right-hand side cancel. By virtue of the Euler equation, the remaining term $\frac{\partial L}{\partial u}$ is equal to p' and we obtain

$$p' = -\frac{\partial H}{\partial u} \quad (19)$$

Next, compute $\frac{\partial H}{\partial p}$. We have

$$\frac{\partial H}{\partial p} = p\frac{\partial \phi}{\partial p} + \phi - \frac{\partial L}{\partial \phi}\frac{\partial \phi}{\partial p}$$

By definition of p, the first and the third term in the right-hand side cancel, and by definition of ϕ ($\phi = u'$) we have

$$u' = \frac{\partial H}{\partial p} \tag{20}$$

The system (19), (20) is called the canonic system, it is remarkable symmetric.

Introducing $2n$ dimensional vector (u, p) of the variables, we combine the equations (19) and (20) as

$$\frac{d}{dx}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \nabla H, \quad \nabla H = \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial p} \end{pmatrix} H \tag{21}$$

1.4 The first integrals through the Hamiltonian

The expressions for the first integrals become very transparent when expressed through impulses the Hamiltonian. If the Lagrangian is independent of u_i', the Hamiltonian is also independent of it. The first integral is $\frac{\partial L}{\partial u_i'} =$ constant. In our notations, this first integral reads

$$p_i = \text{ constant}, \quad \text{ if } H = \text{ constant}(u_i)$$

Similarly, we observe that

$$u_i = \text{ constant}, \quad \text{ if } H = \text{ constant}(p_i)$$

Lagrangian is independent of x

Theorem 1.1 (Constancy of Hamiltionian) If $F = F(u, u')$, equation (**??**) has the first integral:

$$\hat{H}(u, p) = \text{ constant} \tag{22}$$

Proof:

Compute the derivative of the Hamiltonian at the mininizer

$$\frac{d}{dx}H(x, u, p) = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial u}u' + \frac{\partial H}{\partial p}p' = \frac{\partial H}{\partial x}$$

because of equalities (19), $u' = \frac{\partial H}{\partial p}$ and (20), $p' = -\frac{\partial H}{\partial u}$. If Lagrangian does not depend on x, the Hamiltonian is independent of x as well, $\frac{\partial H}{\partial x} = 0$ and we arrive at (22)

In Lagrangian mechanics, the Hamiltonian H is equal to the sum of kinetic and potential energy, $H = T + V$ where $\dot{q}$ is expressed through p, and q, $\dot{q} = R(q)^{-1}p$

$$H(q,p) = \frac{1}{2}p^T R^{-1}p + \frac{\partial}{\partial q}(p^T R^{-1} p + V)$$

Here, we use the fact, that the kinetic energy is a second degree homogeneous function of $\dot{q}$, which implies the equalities

$$\dot{q}^T \frac{\partial T}{\partial \dot{q}} = \dot{q}^T R(q)\dot{q} = 2T$$

Example 1.2 Compute the Hamiltonian and canonic equations for the system in the previous example.

We have

$$L = \frac{1}{2}(a(x)u'^2 + b(x)u^2) = \frac{1}{2}\left(\frac{1}{a(x)}p^2 + b(x)u^2\right)$$

then the Hamiltonian is

$$H = p\left(\frac{p}{a}\right) - L = \frac{1}{2}\left(\frac{1}{a(x)}p^2 - b(x)u^2\right)$$

and the canonic system is

$$\frac{\partial H}{\partial u} = -b(x)u = -p', \quad \frac{\partial H}{\partial p} = \frac{1}{a(x)}p = u'$$

which coincides with the previous example.

1.5 Hamiltonian for geometric optics

The results of study of geometric optics (Section **??**) can be conveniently presented using Hamiltonian. It is convenient to introduce the slowness $w(x,y) = \frac{1}{v(x,y)}$ - reciprocal to the speed v. Then the Lagrangian for the geometric optic problem is

$$L(x,y,y') = w\sqrt{1+(y')^2} \quad y' > 0.$$

Canonic system To find a canonic system, we use the outlined procedure: Define a variable p dual to $y(x)$ by the relation $p = \frac{\partial L}{\partial y'}$

$$p = \frac{wy'}{\sqrt{1+(y')^2}}.$$

Solving for y', we obtain first canonic equation:

$$y' = \frac{p}{\sqrt{w^2 - p^2}} = \phi(x,y,p), \tag{23}$$

Excluding y' from the expression for L,

$$L(x, y, \phi) = L_*(w(x, y), p) = \frac{w^2}{\sqrt{w^2 - p^2}}.$$

and recalling the representation for the solution y of the Euler equation

$$p' = \frac{\partial L}{\partial y} = \frac{\partial L_*}{\partial w} \frac{dw}{dy}$$

we obtain the second canonic equation:

$$p' = -\frac{w}{\sqrt{w^2 - p^2}} \frac{dw}{dy} \tag{24}$$

Hamiltonian Hamiltonian $H = p\phi - L_*(x, y, p)$ can be simplified to the form

$$H = -\sqrt{w^2 - p^2}$$

It satisfies the remarkably symmetric relation

$$H^2 + p^2 = w^2$$

that contains the whole information about the geometric optic problem. The elegancy of this relation should be compared with messy straightforward calculations that we previously did. The geometric sense of the last formula becomes clear if we denote as α the angle of declination of the optimal trajectory to OX axis; then $y' = \tan \alpha$, and (see (**??**))

$$p = \psi(x, y) \sin \alpha, \quad H = -\psi(x, y) \cos \alpha.$$

2 Introduction to Lagrangian mechanics

Leibnitz and Mautoperie suggested that any motion of a system of particles minimizes a functional of action; later Lagrange came up with the exact definition of that action: the functional that has the Newtonian laws of motion as its Euler equation. The question whether the action reaches the true minimum is complicated: Generally, it does not; Nature is more sophisticated and diverse than it was expected. We will show that the true motion of particles settles for a local minimum or even a saddle pint of action' each stationary point of the functional correspond to a motion with Newtonian forces. As a result of realizability of local minima, there are many ways of motion and multiple equilibria of particle system which make our world so beautiful and unexpected (the picture of the rock). The variational principles remain the abstract and economic way to describe Nature but one should be careful in proclaiming the ultimate goal of Universe.

2.1 Stationary Action Principle

Lagrange observed that the second Newton's law for the motion of a particle,

$$m\ddot{x} = f(x)$$

can be viewed as the Euler equation to the variational problem

$$\min_{x(t)} \int_{t_0}^{t_f} \left(\frac{1}{2} m\dot{x}^2 - V(x) \right) dx$$

where V is the negative of antiderivative (potential) of the force f.

$$V = -\int f(x)dx$$

The minimizing quantity – the difference between kinetic and potential energy – is called *action*; The Newton equation for a particle is the Euler equations.

In the stated form, the principle is applicable to any system of free interacting particles; one just need to specify the form of potential energy to obtain the Newtonian motion.

Example 2.1 (Central forces) For example, the problem of celestial mechanics deals with system bounded by gravitational forces f_{ij} acting between any pair of masses m_i and m_j and equal to

$$f_{ij} = \gamma \frac{m_i m_j}{|r_i - r_j|^3} (r_i - r_j)$$

where vectors r_i define coordinates of the masses m_i as follows $r_i = (x_i, y_i, z_j)$. The corresponding potential V for the n-masses system is

$$V = -\frac{1}{2} \sum_{i,j}^{N} \gamma \frac{m_i m_j}{|r_i - r_j|}$$

where γ is Newtonian gravitational constant. The kinetic energy T is the sum of kinetic energies of the particles

$$T = \frac{1}{2} \sum_{i}^{N} m_i \dot{r}_i^2$$

The motion corresponds to the stationary value to the Lagrangian $L = T - V$, or the system of N vectorial Euler equations

$$m_i \ddot{r}_i - \sum_{j}^{N} \gamma \frac{m_i m_j}{|r_i - r_j|^3} (r_i - r_j) = 0$$

for N vector-function $r_i(t)$.

Since the Lagrangian is independent of time t, the first integral (**??**) exist

$$T + V = \text{ constant}$$

which corresponds to the conservation of the whole energy of the system.

Later in Section **??**, we will find other first integrals of this system and comment about properties of its solution.

Example 2.2 (Spring-mass system) Consider the sequence of masses Consider the sequence of masses lying on an axis with coordinates $m_1, \ldots, m_n$ lying on an axis with coordinates $x_1, \ldots, x_n$ joined by the sequence of springs between two sequential masses. Each spring generate force f_i proportional to $x_i - x_{i+1}$ where $x_i - x_{i+1} - l_i$ is the distance between the masses and l_i correspond to the resting spring.

Let us derive the equations of motion of this system. The kinetic energy T of the system is equal to the sum of kinetic energies of the masses,

$$T = \frac{1}{2} m(\dot{x_1} + \ldots + \dot{x_n})$$

the potential energy V is the sum of energies of all springs, or

$$V = \frac{1}{2} C_1 (x_2 - x_1)^2 + \ldots + \frac{1}{2} C_{n-1} (x_n - x_{n-1})^2$$

The Lagrangian $L = T - V$ correspond to n differential equations

$$\begin{aligned}
m_1 \ddot{x}_1 \; &+ \; C_1(x_1 - x_2) = 0 \\
m_2 \ddot{x}_2 \; &+ \; C_2(x_2 - x_3) - C_1(x_1 - x_2) = 0 \\
\ldots \quad & \;\; \ldots \\
m_n \ddot{x}_n \; &- \; C_{n-1}(x_{n-1} - x_n) = 0
\end{aligned}$$

or in vector form

$$M\ddot{x} = P^T C P x$$

where $x = (m_1, \ldots, x_n)$ is the vector of displacements, M is the $n \times n$ diagonal matrix of masses, V is the $(n-1) \times (n-1)$ diagonal matrix of stiffness,

$$M = \begin{pmatrix} m_1 & 0 & \ldots & 0 \\ 0 & m_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & m_n \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & C_{n-1} \end{pmatrix}$$

and P is the $n \times (n-1)$ matrix that corresponds to the operation of difference,

$$P = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

When the masses and the springs are identical, $m_1 = \ldots = m_n = m$ and $C_1 = \ldots = C_{n-1} = C$, the system simplifies to

$$\begin{aligned} m_1\ddot{x}_1 &+ C(x_1 - x_2) = 0 \\ m_2\ddot{x}_2 &+ C(-x_1 + 2x_2 - x_3) = 0 \\ \ldots & \quad \ldots \\ m_n\ddot{x}_n &- C(x_{n-1} - x_n) = 0 \end{aligned}$$

or in vector form,

$$\ddot{x} + kP_2 x = 0$$

where $k = \frac{C}{m}$ is the positive parameter, and $P_2 = P^T P$ is the $n \times n$ matrix of second differences,

$$P_2 = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

2.2 Generalized coordinates

The Lagrangian concept allows for obtaining equations of motion of a constrained system. In this case, the kinetic and potential energy must be defined as a function of *generalized coordinates* that describes degrees of freedom of motion consistent with the constraints. The constraints are be accounted either by Lagrange multipliers or directly, by introducing *generalized coordinates*. If a particle can move along a surface, one can introduced coordinates on this surface and allow the motion only along these coordinates.

The particles can move along the generalized coordinates q_i. Their number corresponds to the allowed degrees of freedom. The position x allowed by constraints becomes $x(q)$. The speed $\dot{x}$ becomes a linear form of $\dot{q}$

$$\dot{x} = \sum \left(\frac{\partial x}{\partial q_i} \dot{q}_i \right)$$

For example, a particle can move along the circle of the radius R, the generalized coordinate will be an angle θ which determines the position $x_1 = R\cos\theta$, $x_2 = R\sin\theta$ at this circle and its speed becomes

$$\dot{x}_1 = -R\dot{\theta}\sin\theta, \quad x\dot{x}_2 = R\dot{\theta}\cos\theta$$

This system has only one degree of freedom, because fixation of one parameter θ completely defines the position of a point.

When the motion is written in terms of generalized coordinates, the constraints are automatically satisfied. Let us trace equations of Lagrangian mechanics in the generalized coordinates. It is needed to represent the potential and kinetic energies in these terms. The potential energy $V(x)$ is straightly rewritten as $W(q) = V(x(q))$ and the kinetic energy $T(\dot{x}) = \sum_i m_i \dot{x}_i^2$ becomes a quadratic form of derivatives of generalized coordinates $\dot{\boldsymbol{q}}$

$$T(\dot{x}) = \sum_i m_i x_i^2 = \dot{\boldsymbol{q}}^T R(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

where the symmetric nonnegative matrix R is equal to

$$R = \{R_{ij}\}, \quad R_{ij} = \left(\frac{\partial T}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial q_i} \right)^T \left(\frac{\partial T}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial q_j} \right)$$

Notice that $T_q(\dot{q})$ is a homogeneous quadratic function of $\dot{q}$, $T_q(k\dot{q}) = k^2 T_q(\dot{q})$ and therefore

$$\frac{\partial}{\partial \dot{\boldsymbol{q}}} T_q(\boldsymbol{q}, \dot{\boldsymbol{q}}) \cdot \dot{\boldsymbol{q}} = 2T_q(\boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{25}$$

the variational problem that correspond to minimal action with respect to generalized coordinates becomes

$$\min_{\boldsymbol{q}} \int_{t_0}^{t_1} (T_q - V_q) dt \tag{26}$$

Because potential energy V does not depend on $\dot{q}$, the Euler equations have the form

$$\frac{d}{dt} \frac{\partial T_q}{\partial \dot{\boldsymbol{q}}} - \frac{\partial}{\partial \boldsymbol{q}} (T_q - V_q) = 0 \tag{27}$$

which is similar to the form of unrestricted motion.

The analogy can be continued. When the Lagrangian is independent of t the system is called *conservative*, In this case, the Euler equation assumes the first integral in the form (use (25))

$$\dot{\boldsymbol{q}} \frac{\partial T_q}{\partial \dot{\boldsymbol{q}}} - (T_q - V_q) = T_q + V_q = \text{constant}(t) \tag{28}$$

Here, $\Pi(q, \dot{q}) = T_q + V_q$ is the whole energy of a mechanical system; it is preserved along the trajectory.

If the generalized impulses $p = R\dot{q}$ are used instead of derivatives $\dot{q}$ of weald coordinates, the whole energy becomes the hamiltonian

$$\Pi(q, \dot{q}) = H(q, p) = p^T R(q)^{-1} p + V_q$$

The generalized coordinates help to formulate differential equations of motion of constrained system. Consider several examples

Example 2.3 (Isochrone) Consider a motion of a heavy mass along the cycloid:

$$x = \theta - \cos\theta, \quad y = \sin\theta$$

To derive the equation of motion, we write down the kinetic T and potential V energy of the mass m, using $q = \theta$ as a generalized coordinate. We have

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = m(1 + \sin\theta)\dot{\theta}^2$$

and $V = my = -m\sin\theta$.

The Lagrangian

$$L = T - V = m(1 + \sin\theta)\dot{\theta}^2 + m\sin\theta$$

allows to derive Euler equation

$$S(\theta, \dot{\theta}) = \frac{d}{dt}\left((1 + \sin\theta)\frac{d\theta}{dt}\right) - \cos\theta = 0.$$

which solution is

$$\theta(t) = \arccos(C_1 \sin t + C_2 \cos t)$$

where C_1 and C_2 are constant of integration. One can check that $\theta(t)$ is 2π-periodic for all values of C_1 and C_2. This explains the name "isochrone" given to the cycloid before it was found that this curve is also the brachistochrone (see Section **??**)

Example 2.4 (Winding around a circle) Describe the motion of a mass m tied to a cylinder of radius R by a rope that winds around it when the mass evolves around the cylinder. Assume that the thickness of the rope is negligible small comparing with the radius R, and neglect the gravity.

It is convenient to use the polar coordinate system with the center at the center of the cylinder. Let us compose the Lagrangian. The potential energy is zero, and the kinetic energy is

$$L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}m\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)^2 + \frac{1}{2}m\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)^2$$
$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

The coordinates $r(t)$ and $\theta(t)$ are algebraically connected by Pythagorean relation $R^2 + l(t)^2 = r(t)^2$ at each time instance t. Here $l(t)$ is the part of the rope that is not winded yet; it is expressed through the angle $\theta(t)$ and the initial length l_0 of the rope, $l(t) = l_0 - R\theta(t)$. We obtain

$$(l_0 - R\theta(t))^2 = r(t)^2 - R^2 \quad \forall t \in [0, t_{\text{final}}],$$

and observe that the time of winding t_{final} is finite. The trajectory $r(\theta)$ is a spiral.

The obtained relation allows for linking of $\dot{r}$ and $\dot{\theta}$. We differentiate it and obtain

$$r\dot{r} = -R(l_0 - R\theta(t))\dot{\theta} = -R(\sqrt{r^2 - R^2}\dot{\theta}$$

or

$$\dot{\theta} = -\frac{\dot{l}}{R} = -\frac{r\dot{r}}{R\sqrt{r^2 - R^2}}$$

The Lagrangian becomes

$$L(r, \dot{r}) = \frac{1}{2} m\dot{r}^2 \left(1 + \frac{r^4}{R^2(r^2 - R^2)}\right)$$

Its first integral

$$\frac{1}{2} m\dot{r}^2 \left(1 + \frac{r^4}{R^2(r^2 - R^2)}\right) = C$$

shows the dependence of the speed $\dot{r}$ on the coordinate r. It can be integrated in a quadratures, leading to the solution

$$t(r) = C_1 \int_{r_0}^{r} \sqrt{\frac{r^2 - R^2}{r^4 + R^2 r^2 - R^4}} dx$$

The two constants r_0 and C_1 are determined from the initial conditions.

The first integral allows us to visualize the trajectory by plotting $\dot{r}$ versus r. Such graph is called the phase portrait of the trajectory.

2.3 More examples: Two degrees of freedom.

Example 2.5 (Move through a funnel) Consider the motion of a heavy particle through a vertical funnel. The axisymmetric funnel is described by the equation $z = \phi(r)$ in cylindrical coordinate system. The potential energy of the particle is proportional to z, $V = -mgz = -mg\phi(r)$ The kinetic energy is

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right)$$

or, accounting that the point moves along the funnel,

$$T = \frac{1}{2} m \left((1 + \phi'^2)\dot{r}^2 + r^2\dot{\theta}^2\right).$$

The Lagrangian

$$L = T - V = \frac{1}{2} m \left((1 + \phi'^2)\dot{r}^2 + r^2\dot{\theta}^2\right) + mg\phi(r)$$

is independent of the time t and the angle θ, therefore two first integrals exist:

$$\frac{\partial L}{\partial \dot{\theta}} = \mu \quad \Rightarrow \dot{\theta} = \frac{\mu}{r^2}$$

and

$$T + V = \frac{1}{2} m \left((1 + \phi'^2)\dot{r}^2 + r^2\dot{\theta}^2\right) - mg\phi(r) = \Pi$$

The second can be simplified by excluding $\dot\theta$ using the first,

$$\Pi = \frac{1}{2} m \left((1+\phi'^2(r))\dot r^2 + \frac{\mu^2}{r^2} - g\phi(r) \right)$$

Here, the constants Π and μ can be defined from the initial conditions. They represent, respectively, the whole energy of the system and the angular momentum; these quantities are conserved along the trajectory. These integrals alone allow for integration of the system, without computing the Euler equations. Solving for $\dot r$, we find

$$(\dot{r^2})^2 = 2\frac{\left(\frac{2\Pi}{m} + g\phi(r)\right) r^2 - \mu^2}{1+\phi'^2(r)}$$

Consequently, we can find $r(t)$ and $\theta(t)$.

Remark 2.1 A constant value $\dot\theta(t)$ constant value of $r(t) = r_0$ corresponds to a periodic trajectory which is defined by the initial energy, angular momentum, and the shape $\phi(r)$ of the funnel, and satisfies the equation

$$\frac{\mu^2}{r_0^2} - g\phi(r_0) = \frac{2\Pi}{m}$$

This equation does not necessary has a solution. Physically speaking, a heavy particle can either evolves around the funnel, or falls down it.

Example 2.6 (Three-dimensional pendulum) A heavy mass is attached to a hitch by a rod of unit length. Describe the motion of the mass. Since the mass moves along the spherical surface, we introduce a spherical coordinate system with the center at the hitch. The coordinates of mass are expressed through two spherical angles ϕ and θ which are the generalized coordinates. We compute

$$T = \dot\phi^2 + \dot\theta^2 \cos\phi$$

and

$$V = g\cos\phi$$

Two conservation laws follows

$$\dot\theta\cos\phi = \mu \tag{29}$$

(conservation of angular momentum) and

$$m(\dot\phi^2 + \dot\theta^2\cos\phi) + g\cos\phi = \Pi \tag{30}$$

(conservation of energy)

The oscillations are described by these two first-order equations for ϕ and θ. The reader is encouraged to use Maple to model the motion.

Two special cases are immediately recognized. When $\mu = 0$, the pendulum oscillates in a plane, $\theta(t) = \theta_0$, and $\dot\theta = 0$. The Euler equation for ϕ becomes

$$m\ddot\phi + g\sin\phi = 0$$

This is the equation for a plane pendulum. The angle $\phi(t)$ is a periodic function of time, the period depends on the magnitude of the oscillations. For small θ, the equation becomes equation of linear oscillator.

When $\phi(t) = \phi_0 = \text{ constant}$, the pendulum oscillates around a horizontal circle. In this case, the speed of the pendulum is constant (see (29)) and the generalized coordinate – the angle θ is

$$\theta = \frac{\mu}{\cos\phi_0} t + \theta_0$$

The motion is periodic with the period

$$T = \frac{2\pi\cos\phi_0}{\mu}$$

Example 2.7 (Two-link pendulum) This example illustrates that the Euler equations for generalized coordinates are similar for the simplest Newton equation $m\ddot{x} = f$ but m becomes a non-diagonal matrix.

Consider the motion of two masses sequentially joined by two rigid rods. The first rod of length l_1 is attached to a hitch and can evolve around it and it has a mass m_1 on its other end. The second rod of the length l_2 is attached to this mass at one end and can evolve around it, and has a mass m_2 at its other end. Let us derive equation of motion of this system in the constant gravitational field.

The motion is expressed in terms of Cartesian coordinates of the masses x_1, y_1 and x_2, y_2. We place the origin in the point of the hitch: This is the natural stable point of the system. The distances between the hitch and the first mass, and between two masses are fixed,

$$l_1 = x_1^2 + y_1^2, \quad l_2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

which reduces the initial four degrees of freedom to two. The constraints are satisfied if we introduce two generalized coordinates: two angles θ_1 and θ_2 of the corresponding rods to the vertical, assuming

$$\begin{aligned} x_1 &= l_1\cos\theta_1, \quad y_1 = l_1\sin\theta_1 \\ x_2 &= l_1\cos\theta_1 + l_2\cos\theta_2, \quad y_2 = l_1\sin\theta_1 + l_2\sin\theta_2. \end{aligned}$$

The state of the system is defined by the angles θ_1 and θ_2.

The potential energy $V(\theta_1, \theta_2)$ is equal to the sum of vertical coordinates of the masses multiplied by the masses,

$$V(\theta_1, \theta_2) = m_1 y_1 + m_2 y_2 = m_1 l_1 \cos\theta_1 + m_2(l_1\cos\theta_1 + l_2\cos\theta_2).$$

The kinetic energy $T = T_1 + T_2$ is the sum of the kinetic energy of two masses:

$$T_1 = \frac{1}{2}m_1(x_1^2 + y_1^2) = \frac{1}{2}m_1\left((-l_1\dot{\theta}_1\sin\theta_1)^2 + (l_1\dot{\theta}_1\cos\theta_1)^2\right) = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2$$

and similarly

$$T_2 = \frac{1}{2}m_2\left((-l_1\dot\theta_1\sin\theta_1 - l_2\dot\theta_2\sin\theta_2)^2 + (l_1\dot\theta_1\cos\theta_1 + l_2\dot\theta_2\cos\theta_2)^2\right)$$
$$= \frac{1}{2}m_2\left(l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2l_1l_2\cos(\theta_1-\theta_2)\dot\theta_1\dot\theta_2\right)$$

Combining these expression, we find

$$T = \frac{1}{2}\dot\theta^T R(\theta_1,\theta_2)\dot\theta$$

where θ is a vector of generalized coordinates$\theta = (\theta_1,\theta_2)^T$, and

$$R(\theta_1,\theta_2) = \begin{pmatrix} (m_2+m_1)\,l_1^2 & m_2l_1l_2\cos(\theta_1-\theta_2) \\ m_2l_1l_2\cos(\theta_1-\theta_2) & m_2l_2^2 \end{pmatrix}$$

is the inertia matrix for the generalized coordinates. The Lagrangian is composed as

$$L = T_1 + T_2 - V$$

Now we immediately derive the equations (27) for the motion:

$$\begin{aligned} s_1 \;&=\; \frac{d}{dt}\left(m_1l_1^2\dot\theta_1 + m_2l_2^2\dot\theta_1 + 2m_2l_1l_2\dot\theta_1\cos(\theta_1-\theta_2)\right) \\ &+\; 2m_2l_1l_2\sin(\theta_1-\theta_2)\dot\theta_1\dot\theta_2 + (m_1l_1+m_2l_2)\sin\theta_1 = 0 \\ s_2 \;&=\; \frac{d}{dt}\left(m_2l_2^2\dot\theta_2 + l_1l_2\cos(\theta_1-\theta_2)\dot\theta_1\right) - 2l_1l_2\sin(\theta_1-\theta_2)\dot\theta_1\dot\theta_2 \\ &-\; m_2l_2\sin\theta_2 = 0 \end{aligned}$$

and notice that the whole energy T_1+T_2+V is constant at all time.

The linearized equations of motion can be derived in an additional assumption $|\theta_i| \ll 1, |\dot\theta_i| \ll 1, i = 1,2.$ they are

$$sl_1 = \left(m_1l_1^2 + m_2l_2^2\right)\ddot\theta_1 \;+\; l_1l_2\ddot\theta_2 + (m_1l_1+m_2l_2)\theta_1 = 0$$
$$sl_2 = m_2l_1l_2\ddot\theta_1 \;+\; m_2l_2^2\ddot\theta_2 - m_2l_2\theta_2 = 0$$

or, in the vector form

$$M\ddot\theta + C\theta = 0$$

where

$$M = \begin{pmatrix} m_1l_1^2 + m_2l_2^2 & m_2l_1l_2 \\ m_2l_1l_2 & m_2l_2^2 \end{pmatrix}, \quad C = \begin{pmatrix} m_1l_1 + m_2l_2 & 0 \\ 0 & m_2l_2 \end{pmatrix}$$

Notice that the matrix M that plays the role of the masses show the inertial elements and it not diagonal but symmetric. The matrix C shows the stiffness of the system. The solution is given by the vector formula

$$\theta(t) = A_1\exp(iBt) + A_2\exp(-iBt), \quad B = (M^{-1}C)^{\frac{1}{2}}$$