Geometric Optics

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1.1 Geometric optics problem.

A half of century before the calculus of variation was invented, Fermat suggested that light propagates along the trajectory which minimizes the time of travel between the source with coordinates (a, A) and the observer with coordinates (b, B). The Fermat principle implies that light travels along straight lines when the medium is homogeneous and along curved trajectories in an inhomogeneous medium in which the speed $v(x, y)$ of light depends on the position. The exactly same problem – minimization of the travel's time – can be formulated as the best route for a cross-country runner; the speed depends on the type of the terrains the runner crosses and is a function of the position. This problem is called the problem of geometric optic.

In order to formulate the problem of geometric optics, consider a trajectory in a plane, call the coordinates of the initial and final point of the trajectory (a, A) and (b, B), respectively, assuming that $a < b$ and call the optimal trajectory $y(x)$ thereby assuming that the optimal route is a graph of a function. The time T of travel can be found from the relation $v = \frac{ds}{dt}$ where $ds = \sqrt{1 + y'^2}dx$ is the infinitesimal length along the trajectory $y(x)$, or

$$dt = \frac{ds}{v(x, y)} = \frac{\sqrt{1 + y'^2}}{v(x, y)}dx$$

Then,

$$T = \int_a^b dt = \int_a^b \frac{\sqrt{1 + y'^2}}{v(x, y)}dx$$

Consider minimization of travel time T by the trajectory. The corresponding Lagrangian has the form

$$F(y, y') = \psi(y)\sqrt{1 + y'^2}, \quad \psi(x, y) = \frac{1}{v(x, y)};$$

and Euler equaltion is

$$\frac{d}{dx}\left(\psi(x, y)\frac{y'}{\sqrt{1 + y'^2}}\right) - \frac{\partial \psi(x, y)}{\partial y}\sqrt{1 + y'^2} = 0, \quad y(a) = y_a, \ y(b) = y_b$$

Assume that the medium is layered and the speed $v(y) = \frac{1}{\psi(y)}$ of travel varies only along the y axes. Then, the problem allows for the first integral, (see (??))

$$\psi(y)\frac{y'^2}{\sqrt{1 + y'^2}} - \psi(y)\sqrt{1 + y'^2} = c$$

which simpleifies to

$$\psi(y) = -c\sqrt{1 + y'^2} \tag{1}$$

Solving for y', we obtain the equation with separated variables

$$\frac{dy}{dx} = \pm \frac{\sqrt{c^2 \psi^2(y) - 1}}{c}$$

with the solution

$$x = \pm \Phi(u) = \int \frac{c\, dy}{\sqrt{\psi^2(y) - c^2}} \tag{2}$$

Notice that equation (1) allows for a geometric interpretation: Derivative y' defines the angle α of inclination of the optimal trajectory, $y' = \tan \alpha$. In terms of α, the equation (1) assumes the form

$$\psi(y)\, \cos \alpha = c \tag{3}$$

which shows that the angle of the optimal trajectory varies with the speed $v = \frac{1}{\psi}$ of the signal in the media. The optimal trajectory is bent and directed into the domain where the speed is higher.

1.2   Snell's law of refraction

Assume that the speed of the signal in medium is piecewise constant; it changes when $y = y_0$ and the speed v jumps from v_+ to v_-, as it happens on the boundary between air and water,

$$v(y) = \begin{cases} v_+ & \text{if } y > y_0 \\ v_- & \text{if } y < y_0 \end{cases}$$

Let us find what happens with an optimal trajectory. Weierstrass-Erdman condition are written in the form

$$\left[\frac{y'}{v\sqrt{1 + y'^2}} \right]_-^+ = 0$$

Recall that $y' = \tan \alpha$ where α is the angle of inclination of the trajectory to the axis OX, then $\frac{y'}{\sqrt{1+y'^2}} = \sin \alpha$ and we arrive at the refraction law called Snell's law of refraction

$$\frac{\sin \alpha_+}{v_+} = \frac{\sin \alpha_-}{v_-}$$

Refraction: Snell's law   Assume the media has piece-wise constant properties, speed $v = 1/\psi$ is piece-wise constant $v = v_1$ in Ω_1 and $v = v_2$ in Ω_2; denote the curve where the speed changes its value by $y = z(x)$. Let us derive the refraction law. The variations of the extremal $y(x)$ on the boundary $z(x)$ can be expressed through the angle θ to the normal to this curve

$$\delta x = \sin \theta, \quad \delta y = \cos \theta$$

Substitute the obtain expressions into the Weierstrass-Erdman condition (**??**) and obtain the refraction law

$$[\psi(\sin\alpha\cos\theta - \cos\alpha\sin\theta)]_-^+ = [\psi\sin(\alpha - \theta)]_-^+ = 0$$

Finally, recall that $\psi = \frac{1}{v}$ and rewrite it in the conventional form (Snell's law)

$$\frac{v_1}{v_2} = \frac{\sin\gamma_1}{\sin\gamma_2}$$

where $\gamma_1 = \alpha_1 - \theta$ and $\gamma_2 = \alpha_2 - \theta$ are, respectively, the angles between the normal to the surface of division and the incoming and the refracted rays.

1.3 Brachistochrone

Problem of the Brachistochrone is probably the most famous problem of classical calculus of variation; it is the problem this discipline start with. In 1696 Bernoulli put forward a challenge to all mathematicians asking to solve the problem: Find the curve of the fastest descent (brachistochrone), the trajectory that allows a mass that slides along it without tension under force of gravity to reach the destination point in a minimal time.

To formulate the problem, we use the law of conservation of the total energy – the sum of the potential and kinetic energy is constant in any time instance:

$$\frac{1}{2}mv^2 + mgy = C$$

where $y(x)$ is the vertical coordinate of the sought curve. From this relation, we express the speed v as a function of u

$$v = \sqrt{C - gy}$$

thus reducing the problem to a special case of geometric optics. (Of course the founding fathers of the calculus of variations did not have the luxury of reducing the problem to something simpler because it was the first and only real variational problem known to the time)

Applying the formula (1), we obtain

$$\frac{1}{\sqrt{C - gy}} = \sqrt{1 + y'^2}$$

and

$$x = \int \frac{\sqrt{y - y_0}}{\sqrt{2a - (y - y_0)}} dy$$

To compute the quadrature, we change the variable in the integral:

$$y = y_0 + 2a\sin^2\frac{\theta}{2}, \quad dy = 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

and find

$$x = 2a \int \sin^2 \frac{\theta}{2} d\theta = a(\theta - \sin \theta) + x_0$$

The optimal trajectory is a parametric curve

$$\begin{aligned} x &= x_0 + a(\theta - \sin \theta), \\ y &= y_0 + a(1 - \cos \theta), \end{aligned} \tag{4}$$

We recognize the equation of the cycloid in (4). Recall that cycloid is a curve generated by a motion of a fixed point on a circumference of the radius a which rolls on the given line $y - y_0$.

Remark 1.1 The obtained solution was formulated in a strange for modern mathematics terms: "Brachistochrone is isochrone." Isochrone was another name for the cycloid; the name refers to a remarkable property of it found shortly before the discovery of brachistochrone: The period of oscillation of a heavy mass that slides along a cycloid is independent of its magnitude.

Notice that brachistochrone is in fact an optimal design problem: the trajectory must be chosen by a designer to minimize the time of travel.

1.4 Minimal surface of revolution

Another classical example of design problem solved by variational methods is the problem of minimal surface. Here, we formulate is for the surface of revolution: Minimize the area of the surface of revolution supported by two circles. According to the calculus, the area J of the surface is

$$J = \pi \int_0^a r\sqrt{1 + r'^2} \, dx \tag{5}$$

This problem is again a special case of the geometric optic, corresponding to $\psi(r) = r$. Equation (2) becomes

$$x = \int \frac{dr}{\sqrt{c^2 r^2 - 1}} = \frac{1}{C} \cosh^{-1}(Cr)$$

and we find

$$r(x) = \frac{1}{C} \cosh\left(C(x - x_0)\right) + c_1$$

Assume for clarity that the surface is supported by two equal circles of radius R parted symmetric to OX axis; then $x_0 = 0$ and $c_1 = 0$ and the surface is

$$r = \frac{1}{C} \cosh\left(Cx\right)$$

The extremals $Cr = \cosh(Cx)$ form a C-dependent family of functions varying only by simultaneously rescaling of the variables r and x. The angle towards a point on the curve from the origin is independent on the scale. This implies

that all curves of the family lie inside the triangular region which boundary is defined by a minimal values of that angle, or by the equation $\frac{r}{x} = f'(x)$ of the enveloping curve. This equation reads

$$\frac{\cosh(r)}{r} = \sinh(r)$$

and leads to $r = 1.199678640...$ All extremals lie inside the triangle $\frac{r}{|x|} \leq 1.50887956$.

Analysis of this formula reveals unexpected features: The solution may be either unique, or has two different solutions (in which case, the one with smaller value of the objective functional must be selected) or it may not have solutions at all. The last case looks strange from the common sense viewpoint because the problem of minimal area obviously has a solution.

The defect in our consideration is the following: We tacitly assumed that the minimal surface of revolution is a differentiable curve with finite tangent y' to the axis of revolution. There is another solution (Goldschmidt solution) : Two circles and an infinitesimal bar between them. The objective functional is $I_0 = 2\pi R^2$.

Goldschmidt solution is a limit of a minimizing sequence that geometrically is a sequence of two cones supported by the circles $y(x_0) = R$ and turned toward each other,

$$r_n(x) = \begin{cases} n\left(x - x_0 + \frac{1}{n}\right)R & x_0 - \frac{1}{n} < x \leq x_0 \\ n\left(-x + x_0 - \frac{1}{n}\right)R & -x_0 + \frac{1}{n} > x \geq -x_0 \end{cases}$$

and an infinitesimally thin cylinder $r = \frac{1}{n^2}$ that join them.

When $n \to \infty$, the central part disappears, the cones flatten and become circles. r_u tents to a discontinuous function

$$\lim_{n \to \infty} r_n(x) = \begin{cases} 0, & |x| \leq x_0 \\ R, & |x| = x_0 \end{cases}$$

The values on functional at the minimizing sequence are the areas of the cones, they obviously tend to the area of the circle. The derivative of the minimizing sequence growth indefinitely in the proximity of the endpoints, however, the intregal (1.4) stays finite. Obviously, this minimizer does not belong to the presumed class of twice-differentiable functions.

From geometrical perspective, the problem should be correctly reformulated as the problem for the best parametric curve $[x(t), r(t)]$ then $r' = \tan\alpha$ where α is the angle of inclination to OX axis. The equation (3) that takes the form

$$r \cos\alpha = C$$

admits either the regular solution $r = C \sec\alpha$, $C \neq 0$ which yields to the catenoid (**??**), or the singular solution $C = 0$ and either $r = 0$ or $\alpha = \frac{\pi}{2}$ which yield to Goldschmidt solution.

Geometric optics suggests a physical interpretation of the result: The problem of minimal surface is formally identical to the problem of the quickest path

between two equally distanced from OX-axis points, if the speed $v = 1/r$ is inverse proportional to the distance to the axis OX. The optimal path between the two close-by points lies along the arch of an appropriate catenoid $r = \cosh(Cx)$ that passes through the given end points. In order to cover the distance quicker, the path sags toward the OX-axis where the speed is larger.

The optimal path between two far-away points is different: The traveler goes straight to the OX-axis where the speed is infinite, then is transported instantly (infinitely fast) to the closest to the destination point at the axis, and goes straight to the destination. This "Harry Potter Transportation Strategy" is optimal when two supporting circles are sufficiently far away from each other.

Reformulation of the minimal surface problem   The paradox in the solution was caused by the infinitimally thin connection path between the supports. One way to regularize it is to request that this path has a finite radius everywhere or to impose an additional constrain $r(x) \geq r_0$. With such constraint, the solution splits into a cylinder $r(x) = r_0$ and the catenoid

$$r(x) = \begin{cases} r_0 & \text{if } 0 \leq x \leq s \\ \frac{\cosh(C(x-C_1))}{C} & \text{if } s \leq x \leq a \end{cases}$$

where s and C and C_1 satisfy two constraints:

$$(C, C_1, s) := \left\{ R = \frac{\cosh(C(a - C_1))}{C}, \quad r_0 = \frac{\cosh(C(s - C_1))}{C} \right\}.$$

The minimal area is given by

$$A(s, C, C_1) = 2 \min_{s,C,C_1 \text{as in above}} \left(2\pi r_0 s - \frac{\sinh(Cs) - \sinh(Cx_0)}{C^2} \right)$$

It remains to find (numerically) optimal values os s and C and C_1 that are subject to above constraints. Notice that s can be find from the transversality condition instead of direct minimization of A.