Constrainted problems

March 6, 2013

Contents

1 Introduction

Many variational problems ask for a constrained minimum of a functional. There are several types of constraints that we discuss.

(i) Isoperimetric problem (Section 3). Minimize an integral functional if the other(s) integral functionals must stay constant. The classic example is the Didi problem: maximize the encircled area by a robe of a given length.

$$\min_{y(x)=\in\mathcal{Y},T} \int_0^T y(x)\,dx \quad \text{subject to} \quad \int_0^1 \sqrt{1+(y')^2}\,dx \quad \text{subject to } \phi(u)=0 \quad (1)$$

where

$$\mathcal{Y} = \{y: \ u(0)=0, u(T)=0\}$$

In addition, the problem of minimization of the product, ratio, or superposition of the integrals can be reduced to constrained variational problems, as described in Section 4. Am example is the problem of the principle eigenfrequency that is defined as a ratio between the total potential and kinetic energy of an oscillating body.

(ii) Problem with constraints imposed at each point of the trajectory (Section **??**). The example is a problem of geodesics: minimize the distance of the path between two points if the path everywhere belongs to a given surface $\phi(u)=0$.

$$\min_{u(t)=\mathcal{U}} \int_0^1 \sqrt{1+(u')^2}\,dx \quad \text{subject to } \phi(u)=0 \tag{2}$$

where

$$\mathcal{U} = \{u=(u_1(t),u_2(t),u_3(t)): \ u(0)=A, u(1)=B\}$$

(iii) Problem with differential constraints (Section **??**). The simplest variational problem can be rewrite this way:

$$\min_{u(x),v(x)} \int_0^T F(x,u,v)\,dx \quad \text{subject to } u'=v \tag{3}$$

More general differential constraints $L(u,u')=0$ can be imposed as well. A example is the problem of minimization of the fuel consumption of a vehicle that moves in the given time T between two points A and B, if the rate of fuel spend f defines the speed of the vehicle u' through the differential equation of the system $L(u,u',f)=0$.

$$\min_{f\in\mathcal{F}} \int_0^T f(t)\,dt \quad \text{subject to } L(u,u'f)=0, \ u(0)=A, \ u(T)=B. \tag{4}$$

(iv) Control theory include the differential constraints, and inequality of the set of minimizers called controls. The previous problem becomes a control theory problem if additionally the rate of spend fuel is bounded everywhere. where

$$\mathcal{F} = \{f: \ 0\le f(t)\le f_{\max} \ \forall t\in[0,T]\}$$

In the control theory, the differential constrains are traditionally expressed in Cauchy form as follows

$$L(u, u', f) = 0 \;\Leftrightarrow\; z_i' = g(z_1, ..z_n, u_1, ..u_m), i = 1, \dots, n$$

where $f \in \mathcal{F}$. This canonical form describe a system of ordinary differential equation in a standard fashion. Regrettably, it cannot be generalized to partial differential equations, see discussion in Section**??**.

2 Lagrange multipliers: Vector problem

2.1 Lagrange Multipliers method

Reminding of the technique discussed in calculus, we first consider a finite-dimensional problem of constrained minimum. Namely, we want to find the condition of the minimum:

$$J = \min_x f(x), \quad x \in R^n, \quad f \in C_2(R^n) \tag{5}$$

assuming that m constraints are applied

$$g_i(x_1, \dots x_n) = 0 \quad i = 1, \dots m, \quad m \le n, \tag{6}$$

The vector form of the constraints is

$$g(x) = 0$$

where g is a m-dimensional vector-function of an n-dimensional vector x.

To find the minimum, we add the constraints with the Lagrange multipliers $\boldsymbol{\mu} = (\mu_1, \dots \mu_p)$ and end up with the problem

$$J = \min_x \left[f(x) + \sum_i^m \mu_i g_i(x) \right]$$

The stationary conditions become:

$$\frac{\partial f}{\partial x_k} + \sum_i^m \mu_i \frac{\partial g_i}{\partial x_k} = 0, \quad k = 1, \dots, n$$

or, in the vector form

$$\frac{\partial f}{\partial x} + W \cdot \boldsymbol{\mu} = 0 \tag{7}$$

where the $m \times n$ Jacobian matrix W is

$$W = \frac{\partial g}{\partial x} \quad \text{or, by elements, } W_{nm} = \frac{\partial g_n}{\partial x_m}$$

The system (7) together with the constraints (6) forms a system of $n + p$ equations for $n + p$ unknowns: Components of the vectors x and $\boldsymbol{\mu}$.

Minimax background Consider again the finite-dimensional minimization problem

$$J = \min_{x_1,\dots x_n} F(x_1, \dots x_n) \tag{8}$$

subject to one constraint

$$g(x_1, \dots x_n) = 0 \tag{9}$$

and assume that solutions to (9) exist in a neighborhood of the minimal point. It is easy to see that the described constrained problem is equivalent to the unconstrained problem

$$J_* = \min_{x_1,\dots x_n} \max_{\lambda} \left(F(x_1, \dots x_n) + \lambda g(x_1, \dots x_n)\right) \tag{10}$$

Indeed, the inner maximization gives

$$\max_{\lambda} \lambda g(x_1, \dots x_n) = \begin{cases} \infty & \text{if } g \neq 0 \\ 0 & \text{if } g = 0 \end{cases}$$

because λ can be made arbitrary large or arbitrary small. This possibility forces us to choose such x that delivers equality in (9), otherwise the cost of the problem (10) would be infinite (recall that x "wants" to minimize J_*). By assumption, such x exists. At the other hand, the constrained problem (8)-(9) does not change its cost J if zero (recall, that $\lambda g(x) = 0$) is added to it. Thereby $J = J_*$ and the problem (8) and (9) is equivalent to (10).

Augmented problem If we interchange the sequence of the two extremal operations in (10), we would arrive at the *augmented* problem J_D

$$J \geq J_D(x, \lambda) = \max_{\lambda} \min_{x_1,\dots x_n} \left(F(x_1, \dots x_n) + \lambda g(x_1, \dots x_n)\right) \tag{11}$$

The interchange of max and min- operations preserves the problems cost if $F(x_1, \dots x_n) + \lambda g(x_1, \dots x_n)$ is a convex function of $x_1, \dots x_n$; in this case $J = J_D$. In a general case, we arrive at an inequality $J \geq J_D$ (see the min-max theorem in Section **??**)

The extended Lagrangian J_* depends on $n+1$ variables. The stationary point corresponds to a solution to a system

$$\frac{\partial L}{\partial x_k} = \frac{\partial F}{\partial x_k} + \lambda \frac{\partial g}{\partial x_k} = 0, \quad k = 1, \dots n, \tag{12}$$

$$\frac{\partial L}{\partial \lambda} = g(x) = 0 \tag{13}$$

The procedure is easily generalized for several constrains. In this case, we add each constraint with its own Lagrange multiplier to the minimizing function and arrive at expression (7)

Example Consider the problem

$$J = \min_x \sum_{i=1}^n A_i^2 x_i \quad \text{subject to} \quad \sum_{i=1}^n \frac{1}{x_i - k} = \frac{1}{c}, \quad x_i > k, c > 0, k > 0.$$

Using Lagrange multiplier λ we rewrite it in the form:

$$J_a = \min_x \sum_{i=1}^n A_i^2 x_i + \lambda \left(\sum_{i=1}^n \frac{1}{x_i - k} - \frac{1}{c} \right).$$

From the condition $\frac{\partial J_a}{\partial x} = 0$ we obtain

$$A_i^2 - \frac{\lambda}{(x_i - k)^2} = 0, \quad \text{or} \quad \frac{1}{x_i - k} = \frac{|A_i|}{\sqrt{\lambda}} \quad i = 1, \dots, n.$$

Substitute these values into expression for the constraint and obtain an equation for λ

$$\frac{1}{c} = \sum_{i=1}^n \frac{1}{x_i - k} = \frac{1}{\sqrt{\lambda}} \sum_{i=1}^n |A_i|$$

Solving this equation, we find λ and the minimizer x_i

$$\sqrt{\lambda} = c \sum_{i=1}^n |A_i|, \quad x_i = k + \frac{\sqrt{\lambda}}{|A_i|},$$

as well as the optimal value of J:

$$J = k \sum_{i=1}^n A_i^2 + c \left(\sum_{i=1}^n |A_i| \right)^2$$

Observe, the minimum is a sum of squares of L_2 and L_1 norms of the vector $A = [A_1, \dots, A_n]$.

2.2 Exclusion of Lagrange multipliers and duality

We can exclude the multipliers $\boldsymbol{\mu}$ from the system (7) assuming that the constraints are independent, that is $\mathrm{rank}(W) = m$. We project n-dimensional vector ∇F onto a $n - m$-dimensional subspace allowed by the constraints, and require that this projection is zero. The procedure is as follows.

1. Multiply (7) by W^T:

$$W^T \frac{\partial f}{\partial x} + W^T W \cdot \boldsymbol{\mu} = 0, \tag{14}$$

   Since the constraints are independent, $m \times m$ matrix $W^T W$ is nonsingular, $\det(W^T W) \neq 0$.

2. Find m-dimensional vector of multipliers $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} = -(W^T W)^{-1} W^T \frac{\partial f}{\partial x},$$

3. Substitute the obtained expression for $\boldsymbol{\mu}$ into (7) and obtain:

$$(I - W(W^T W)^{-1} W^T) \frac{\partial f}{\partial x} = 0 \tag{15}$$

Matrix $W(W^T W)^{-1} W^T$ is called the projector to the subspace W. Notice that the rank of the matrix $W(W^T W)^{-1} W^T$ is equal to m; it has p eigenvalues equal to one and $n - m$ eigenvalues equal to zero. Therefore the rank of $I - W(W^T W)^{-1} W^T$ is equal to $n - m$, and the system (15) produces $n - m$ independent optimality conditions. The remaining m conditions are given by the constraints (6): $g_i = 0,\ i = 1, \dots m$. Together these two groups of relations produce n equations for n unknowns $x_1, \dots, x_n$.

Below, we consider several special cases.

Degeneration: No constraints When there is no constraints, $W = 0$, the problem trivially reduces to the unconstrained on, and the necessary condition (15) becomes $\frac{\partial f}{\partial x} = 0$ holds.

Degeneration: $\boldsymbol{n}$ constraints Suppose that we assign n independent constraints. They themselves define vector x and no additional freedom to choose it is left. Let us see what happens with the formula (15) in this case. The rank of the matrix $W(W^T W)^{-1} W^T$ is equal to n, (W^{-1} exists) therefore this matrix-projector is equal to I:

$$W(W^T W)^{-1} W^T = I$$

and the equation (15) becomes a trivial identity. No new condition is produced by (15) in this case, as it should be. The set of admissible values of x shrinks to the point and it is completely defined by the n equations $g(x) = 0$.

One constraint Another special case occurs if only one constraint is imposed; in this case $m = 1$, the Lagrange multiplier $\boldsymbol{\mu}$ becomes a scalar, and the conditions (7) have the form:

$$\frac{\partial f}{\partial x_i} + \mu \frac{\partial g}{\partial x_i} = 0 \quad i = 1, \dots n$$

Solving for μ and excluding it, we obtain $n - 1$ stationary conditions:

$$\frac{\partial f}{\partial x_1} \left(\frac{\partial g}{\partial x_1} \right)^{-1} = \dots = \frac{\partial f}{\partial x_n} \left(\frac{\partial g}{\partial x_n} \right)^{-1} \tag{16}$$

Let us find how does this condition follow from the system (15). This time, W is a $1 \times n$ matrix, or a vector,

$$W = \left[\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}\right]$$

We have:

$$\text{rank } W(W^TW)^{-1}W^T = 1, \quad \text{rank}(I - W(W^TW)^{-1}W^T) = n - 1$$

Matrix $I - W(W^TW)^{-1}W^T$ has $n-1$ eigenvalues equal to one and one zero eigenvalue that corresponds to the eigenvector W. At the other hand, optimality condition (15) states that the vector

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]$$

lies in the null-space of the matrix $I - W(W^TW)^{-1}W^T$ that is vectors $\frac{\partial f}{\partial x}$ and W are parallel. Equation (16) expresses parallelism of these two vectors.

Quadratic function Consider minimization of a quadratic function

$$F = \frac{1}{2}x^TAx + d^Tx$$

subject to linear constraints

$$Bx = \beta$$

where $A > 0$ is a positive definite $n \times n$ matrix, B is a $n \times m$ matrix of constraints, d and β are the n- and m-dimensional vectors, respectively. Here, $W = B$. The optimality conditions consist of m constraints $Bx = \beta$ and $n-m$ linear equations

$$(I - B(B^TB)^{-1}B^T)(Ax + d) = 0$$

2.3 Duality

Let us return to the constraint problem

$$J = \min_x \max_\mu (F(x) + \mu^T g(x))$$

with the stationarity conditions,

$$\nabla F + \mu^T W(x) = 0$$

Instead of excluding μ as is was done before, now we do the opposite: Exclude n-dimensional vector x from n stationarity conditions, solving them for x and thus expressing x through μ: $x = \phi(\mu)$. When this expression is substituted into original problem, the later becomes

$$J_D = \max_\mu \{F(\phi(\mu)) + \mu^T g(\phi(\mu))\};$$

it is called *dual* problem to the original minimization problem.

Notice that the dual problem asks for maximization of the goal function, therefore any admissible vector μ provides the *lower bound* for J_D and, therefore, for J. Recall, that any admissible vector x provides the upper bound for the original minimization problem. Therefore, the pair of the admissible vectors x and u in the corresponding optimization problems give the two-side bounds for the goal function.

Dual form for quadratic problem Consider again minimization of a quadratic. Let us find the dual form for it. We solve the stationarity conditions $Ax+d+B^T\mu$ for x, obtain

$$x = -A^{-1}(d + B^T\mu)$$

and substitute it into the extended problem:

$$J_D = \max_{\mu\in R_m} \left\{ \frac{1}{2}(d^T + \mu^T B)A^{-1}(d + B^T\mu) - \mu^T BA^{-1}(d + B^T\mu) - \mu^T\beta \right\}$$

Simplifying, we obtain

$$J_D = \max_{\mu\in R_m} \left\{ -\frac{1}{2}\mu^T BA^{-1}B^T\mu - \mu^T\beta + \frac{1}{2}d^T A^{-1}d \right\}$$

The dual problem is also a quadratic form over the m dimensional vector of Lagrange multipliers μ; observe that the right-hand-side term β in the constraints in the original problem moves to the sift term in the dual problem. The shift d in the original problem generates an additive term $\frac{1}{2}d^T A^{-1}d$ in the dual one.

2.4 Finite-dimensional variational problem

Consider the optimization problem for a finite-difference system of equations

$$J = \min_{y_1,\ldots,y_N} \sum_i^N f_i(y_i, z_i)$$

where $f_1, \ldots, f_N$ are given value of a function f, $y_1, \ldots, y_N$ is the N-dimensional vector of unknowns, and z_i $i = 2, \ldots, N$ are the finite differences of y_i:

$$z_i = \text{Diff}(y_i) \quad \text{where } \text{Diff}(y_i) = \frac{1}{\Delta}(y_i - y_{i-1}), \quad i = 1, \ldots, N \qquad (17)$$

Assume that the boundary values y_1 and y_n are given and take (17) as constraints. Using Lagrange multiplies $\mu_1, \ldots, \mu_N$ we pass to the augmented function

$$J_a = \min_{y_1,\ldots,y_N;\ z_1,\ldots,z_N} \sum_i^N \left[f_i(y_i, z_i) + \mu_i \left(z_i - \frac{1}{\Delta}(y_i - y_{i-1}) \right) \right]$$

The necessary conditions are:

$$\frac{\partial J_a}{\partial y_i} = \frac{\partial f_i}{\partial y_i} + \frac{1}{\Delta}(-\mu_i + \mu_{i+1}) = 0 \quad 2 = 1, \dots, N-1$$

and

$$\frac{\partial J_a}{\partial z_i} = \frac{\partial f_i}{\partial z_i} + \mu_i = 0 \quad i = 2, \dots, N-1$$

Excluding μ_i from the last equation and substituting their values into the previous one, we obtain the conditions:

$$\frac{\partial J_a}{\partial y_i} = \frac{\partial f_i}{\partial y_i} + \frac{1}{\Delta}\left(\frac{\partial f_i}{\partial z_i} - \frac{\partial f_{i+1}}{\partial z_{i+1}}\right) = 0 \quad i = 2, \dots, N-1$$

or, recalling the definition of the Diff -operator,

$$\text{Diff}\left(\frac{\partial f_{i+1}}{\partial z_{i+1}}\right) - \frac{\partial f_i}{\partial y_i} = 0 \quad z_i = \text{Diff}(y_i) \tag{18}$$

One can see that the obtained necessary conditions have the form of the difference equation of second-order.

Formal passage to differential equation Diff-operator is an approximation of a derivative and the equation (18) is a finite-difference approximation of the Euler equation. When $N \to \infty$,

$$\text{Diff}(y_i) \to \frac{dy}{dx}$$

and we obtain the differential equation of the second order (the Euler equation):

$$\frac{d}{dx}\frac{\partial F}{\partial u'} - \frac{\partial F}{\partial u} = 0$$

for the unknown minimizer – function $y(x)$.

2.5 Inequality constraints

Nonnegative Lagrange multipliers Consider the problem with a constraint in the form of inequality:

$$\min_{x_1, \dots x_n} F(x_1, \dots x_n) \quad \text{subject to } g(x_1, \dots x_n) \leq 0 \tag{19}$$

In order to apply the Lagrangian multipliers technique, we reformulate the constraint:

$$g(x_1, \dots x_n) + v^2 = 0$$

where v is a new auxiliary variable.

The augmented Lagrangian becomes

$$L_*(x, v, \lambda) = f(x) + \lambda g(x) + \lambda v^2$$

and the optimality conditions with respect to v are

$$\frac{\partial L_*}{\partial v} = 2\lambda v = 0 \tag{20}$$

$$\frac{\partial^2 L_*}{\partial v^2} = 2\lambda \geq 0 \tag{21}$$

The second condition requires the nonnegativity of the Lagrange multiplier and the first one states that the multiplier is zero, $\lambda = 0$, if the constraint is satisfied by a strong inequality, $g(x_0) > 0$.

The stationary conditions with respect to x

$$\begin{array}{ll} \nabla f = 0 & \text{if } g \leq 0 \\ \nabla f + \lambda \nabla g = 0 & \text{if } g = 0 \end{array}$$

state that either the minimum correspond to an inactive constraint ($g > 0$) and coincide with the minimum in the corresponding unconstrained problem, or the constraint is active ($g = 0$) and the gradients of f and g are parallel and directed in opposite directions:

$$\frac{\nabla f(x_b) \cdot \nabla g(x_b)}{|\nabla f(x_b)|\,|\nabla g(x_b)|} = -1, \quad x_b : \; g(x_b) = 0$$

In other terms, the projection of $\nabla f(x_b)$ on the subspace orthogonal to $\nabla g(x_b)$ is zero, and the projection of $\nabla f(x)$ on the direction of $\nabla g(x_b)$ is nonpositive.

The necessary conditions can be expressed by a single formula using the notion of infinitesimal variation of x or a differential. Let x_0 be an optimal point, x_{trial} – an admissible (consistent with the constraint) point in an infinitesimal neighborhood of x_0, and $\delta x = x_{\text{trial}} - x_0$. Then the optimality condition becomes

$$\nabla f(x_0) \cdot \delta x \geq 0 \quad \forall \delta x \tag{22}$$

Indeed, in the interior point x_0 ($g(x_0) > 0$) the vector δx is arbitrary, and the condition (22) becomes $\nabla f(x_0) = 0$. In a boundary point x_0 ($g(x_0) = 0$), the admissible points are satisfy the inequality $\nabla g(x_0) \cdot \delta x \leq 0$, the condition (22) follows from (21).

It is easy to see that the described constrained problem is equivalent to the unconstrained problem

$$L_* = \min_{x_1, \dots x_n} \max_{\lambda > 0} \left(F(x_1, \dots x_n) + \lambda g(x_1, \dots x_n) \right) \tag{23}$$

that differs from (11) by the requirement $\lambda > 0$.

Several constraints: Kuhn-Tucker conditions Several inequality constraints are treated similarly. Assume the constraints in the form

$$g_1(x) \leq 0, \dots, g_m(x) \leq 0.$$

The stationarity condition can be expressed through nonnegative Lagrange multipliers

$$\nabla f + \sum_{i=1}^{m} \lambda_i \nabla g_i = 0, \tag{24}$$

where

$$\lambda_i \geq 0, \quad \lambda_i g_i = 0, \quad i = 1, \ldots, m. \tag{25}$$

The minimal point corresponds either to an inner point of the permissible set (all constraints are inactive, $g_i(x_0) < 0$), in which case all Lagrange multipliers λ_i are zero, or to a boundary point where $p \leq m$ constraints are active. Assume for definiteness that the first p constraints are active, that is

$$g_1(x_0) = 0, \quad \ldots, \quad g_p(x_0) = 0. \tag{26}$$

The conditions (25) show that the multiplier λ_i is zero if the ith constrain is inactive, $g_i(x) > 0$. Only active constraints enter the sum in (27), and it becomes

$$\nabla f + \sum_{i=1}^{p} \lambda_i \nabla g_i = 0, \quad \lambda_i > 0, \quad i = 1, \ldots, p. \tag{27}$$

The term $\sum_{i=1}^{p} \lambda_i \nabla g_i(x_0)$ is a cone with the vertex at x_0 stretched on the rays $\nabla g_i(x_0) > 0$, $i = 1, \ldots, p$. The condition (27) requires that the negative of $\nabla f(x_0)$ belongs to that cone.

Alternatively, the optimality condition can be expressed through the admissible vector δx,

$$\nabla f(x_0) \cdot \delta x \geq 0 \tag{28}$$

Assume again that the first p constraints are active, as in (**??**)

$$g_1(x_0) =, \ldots, = g_p(x_0) = 0.$$

In this case, the minimum is given by (28) and the admissible directions of δx satisfy the system of linear inequalities

$$\delta x \cdot \nabla g_i \geq 0, \quad i = 1, \ldots, p. \tag{29}$$

These conditions are called Kuhn-Tucker conditions, see []

3 Isoperimetric problem

3.1 Stationarity conditions

Isoperimetric problem of the calculus of variations is

$$\min_u \int_a^b F(x, u, u')dx \quad \text{subject to} \quad \int_a^b G(x, u, u')dx = 0 \tag{30}$$

Applying the same procedure as in the finite-dimensional problem, we reformulate the problem using Lagrange multiplier λ:

$$\min_u \int_a^b \left[F(x,u,u') + \lambda G(x,u,u')\right] dx \tag{31}$$

To justify the approach, we may look on the finite-dimensional analog of the problem

$$\min_{u_i} \sum_{i=1}^N F_i(u_i, \mathrm{Diff}(u_i)) \quad \text{subject to} \quad \sum_{i=1}^N G_i(u_i, \mathrm{Diff}(u_i)) = 0$$

The Lagrange method is applicable to the last problem which becomes

$$\min_{u_i} \sum_{i=1}^N \left[F_i(u_i, \mathrm{Diff}(u_i)) + \lambda G_i(u_i, \mathrm{Diff}(u_i))\right].$$

Passing to the limit when $N \to \infty$ we arrive at (31).

The procedure of solution is as follows: First, we solve Euler equation for the problem(31)

$$\frac{d}{dx}\frac{\partial}{\partial u'}(F + \lambda G) - \frac{\partial}{\partial u}(F + \lambda G) = 0.$$

Keeping λ undefined and arrive at minimizer $u(x,\lambda)$ which depends on parameter λ. The equation

$$\int_a^b G(x, u(x,\lambda), u'(x,\lambda)) dx = 0$$

defines this parameter.

Remark 3.1 The method assumes that the constraint is consistent with the variation: The variation must be performed upon a class of functions u that satisfy the constraint. Parameter λ has the meaning of the cost for violation of the constraint.

Of course, it is assumed that the constraint can be satisfied for all varied functions that are close to the optimal one. For example, the method is not applicable to the constraint

$$\int_a^b u^2 dx \leq 0$$

because this constraint allows for only one function $u = 0$ and will be violated at any varied trajectory.

3.2 Dido problem revisited

Let us apply the variational technique to Dido Problem discussed in Chapter **??**. It is required to maximize the area A between the OX axes and a positive curve $u(x)$

$$A = \int_a^b u dx \quad u(x) \geq 0 \forall x \in [a,b]$$

assuming that the length L of the curve is given

$$L = \int_a^b \sqrt{1+u'^2}dx$$

and that the beginning and the end of the curve belong to OX-axes: $u(a) = 0$ and $u(b) = 0$. Without lose of generality we assume that $a = 0$ and we have to find b.

The constrained problem has the form

$$J = A + \lambda L = \int_0^b \left(u + \lambda\sqrt{1+u'^2}\right) dx$$

where λ is the Lagrange multiplier.

The Euler equation for the extended Lagrangian is

$$1 - \lambda \frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right)$$

Let us fix λ and find u as a function of x and λ. Integrating, we obtain

$$\lambda \frac{u'}{\sqrt{1+u'^2}} = x - C_1$$

where C_1 is a constant of integration. Solving for $u' = \frac{du}{dx}$, we rewrite the last equation as

$$du = \pm \frac{(x - C_1)dx}{\sqrt{\lambda^2 + (x - C_1)^2}},$$

integrate it:

$$u = \mp\sqrt{\lambda^2 + (x - C_1)^2} + C_2$$

and rewrite the result as

$$(x - C_1)^2 + (u - C_2)^2 = \lambda^2 \tag{32}$$

The extremal is a part of the circle. The constants C_1, C_2 and λ can be found from boundary conditions and the constraints.

To find the length b of the trajectory, we use the transversality condition (**??**):

$$u'\frac{\partial F}{\partial u'} - F|_{x=b} = -\frac{\lambda}{\sqrt{1+u'^2}} - u\Big|_{x=b,u(b)=0} = \frac{\lambda}{\sqrt{1+u'^2}}\Big|_{x=b} = 0$$

which gives $|u'(b)| = \infty$. We have shown that the optimal trajectory approaches OX-axis at the point b perpendicular to it. By symmetry, $|u'(a)| = \infty$ as well which means that the optimal trajectory is the semicircle of the radius λ, symmetric with respect to OX-axis. We find $\lambda = \frac{L}{\pi}$, $C_1 = a + \frac{L}{2\pi}$, and $C_2 = 0$.

3.3 Catenoid

The classical problem of the shape of a heavy chain (catenoid, from Latin *catena*) was considered by Euler ?? using a *variational principle*. It is postulated, that the equilibrium minimizes the potential energy W of the chain

$$W = \int_0^1 g\rho u \, ds = g\rho \int_0^1 u\sqrt{1 + (u')^2} dx$$

defined as the limit of the sum of vertical coordinates of the parts of the chain. Here, ρ is the density of the mass of the chain, ds is the element of its length, x and u are the horizontal and vertical coordinates, respectively. The length of the chain

$$L = \int_0^1 \sqrt{1 + (u')^2} dx$$

and the coordinates of the ends are fixed. Normalizing, we put $g\rho = 1$. Formally, the problem becomes

$$I = \min_{u(x)} \ (W(u) + \lambda L(u)), \quad W(u) + \lambda L(u) = \int_0^1 (u + \lambda)\sqrt{1 + (u')^2} dx$$

The Lagrangian is independent of x and therefore permits the first integral

$$(u + \lambda) \left(\frac{(u')^2}{\sqrt{1 + (u')^2}} - \sqrt{1 + (u')^2} \right) = C$$

that is simplified to

$$\frac{u + \lambda}{\sqrt{1 + (u')^2}} = C.$$

We solve for u'

$$\frac{du}{dx} = \sqrt{\left(\frac{u + \lambda}{C}\right)^2 - 1}$$

integrate

$$x = \ln\left(\lambda + u + \sqrt{\left(\frac{u + \lambda}{C}\right)^2 - 1}\right) - \ln C + x_0$$

and find the extremal $u(x)$

$$u = -C\cosh\left(\frac{x - x_0}{C}\right) + \lambda$$

The equation – the catenoid – defines the shape of a chain; it also gave the name to the hyperbolic cosine.

3.4 Isoperimetric problems for multiple integrals

$$I = \int_\Omega F(x, u, \nabla u) dx.$$

is minimized subject to the *isoperimetric* constraint

$$C = \int_\Omega G(x, u, \nabla u) dx = 0 \tag{33}$$

The problems with isoperimetric constraints are addressed similarly to the one-variable case.

The augmented Lagrangian AI is constructed by adding the constraint C with a Lagrange multiplier λ to the Lgrangian I as follows

$$AI = I + \lambda C = \int_\Omega \left(F(x, u, \nabla u) + \lambda G(x, u, \nabla u)\right) dx. \tag{34}$$

The minimizer u (if exists) satisfies the Euler-Lagrange equation

$$\left(\nabla \cdot \frac{\partial}{\partial(\nabla u)} - \frac{\partial}{\partial u} \right) (F + \lambda G) = 0 \tag{35}$$

and the natural boundary condition

$$n \cdot \frac{\partial}{\partial(\nabla u)} (F + \lambda G) = 0 \quad \text{ on } \partial\Omega. \tag{36}$$

If the main boundary condition $u = \phi$ on $\partial\Omega$ is prescribed, the Euler equation (35) is solved subject to this boundary condition instead of (36).

Solution depends on parameter λ, $u = u(\lambda)$, it is defined from the constraint (33).

Example 3.1 Consider the generalization of the problem of analytic continuation (Example **??**): Find a function $u(x)$ which is equal to a given function ϕ on the boundary $\partial\Omega$ of Ω and minimizes the integral of the Lagrangian $W = \frac{1}{2}|\nabla u|^2$ over the domain Ω. Generalizing the problem, we additionally require to keep the integral equality

$$\int_\Omega \rho\, u\, dx = 0 \tag{37}$$

where $\rho = \rho(x)$ is a weight function. The augmented functional is

$$I = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 + \lambda \rho u \right) dx$$

where λ is the Lagrange multiplier by the constraint (37). The Euler equation is

$$\Delta u - \lambda \rho = 0 \ \text{ in } \Omega, \quad u = \phi \ \text{ on } \partial\Omega$$

We solve this equation using the potential theory [?] and treating λ as a known parameter and obtain $u = u(\lambda)$. Then, we compute integral of $\rho u(\lambda)$ and solve the equation (37) for λ. The potential u has the representation

$$u(x) = \lambda\rho \int_\Omega G(x,\xi)\rho(\xi)d\xi + \int_{\partial\Omega} \frac{\partial G(x,s)}{\partial n}\phi(s)ds$$

where $G(x,\xi)$ is the Green's function for Poisson equation.

Referring to [], we check that

$$G(x,\xi) = \begin{cases} \frac{1}{4\pi}\frac{1}{|x-\xi|} & \text{if } \Omega \subset R^3 \\ -\frac{1}{2\pi}\log|x-\xi| & \text{if } \Omega \subset R^2 \end{cases}. \tag{38}$$

Substituting this expression for u into (37), we find λ in the closed form,

$$\lambda = -\frac{\int_\Omega \rho(x)\left(\int_{\partial\Omega} \frac{\partial G(x,s)}{\partial n}\phi(s)ds\right)dx}{\int_\Omega \rho(x)\left(\int_\Omega G(x,\xi)\rho(\xi)d\xi\right)dx}$$

3.5 Constrained and Incomplete boundary conditions

In some cases, the exact boundary conditions are not available, but only some integrals of them,

$$\int_{\partial\Omega} \psi(u,s)ds = 0. \tag{39}$$

For example, one may not know the exact potential on boundary components, but only the average drop of potential between them. Likewise, one may know the total load on the construction but not the distribution of the load on its boundary. On the other hand, in order to determine the potential from the boundary value problem, the pointwise boundary conditions are needed.

This problem can be addressed by the variational method. We require some point-wise additional boundary conditions that minimize the energy and satisfy the integral condition (39). This way, one obtains natural boundary conditions from the artificially added optimality requirement and the given integral constraints. The correspond to the best/worse case scenarios.

The Lagrange multipliers method is used to formulate the problem quite similar to the previous example. Considering the problem (35) - (36), we add (39) with the Lagrange multiplier μ and obtain the natural boundary condition in the form

$$n \cdot \frac{\partial}{\partial(\nabla u)}(F + \lambda G) + \mu\frac{\partial\psi}{\partial u} = 0 \quad \text{on } \partial\Omega.$$

Example 3.2 (Incomplete boundary conditions for a conducting rectangle) Consider the following problem. A square homogeneous domain Ω with the side a conducts electric current from the left $\partial\Omega_1$ to the right $\partial\Omega_2$ parts of the boundary, while the upper and lower sides $\partial\Omega_3$ and $\partial\Omega_4$ are insulated. The mean potential drop between the left side $\partial\Omega_1$ and the right side $\partial\Omega_2$ is equal to V but the detailed distribution of the potential over the boundary is unknown. We avoid the uncertainty requiring that the boundary potentials would minimize the total energy of the domain. Assuming these boundary conditions, we formulate the variational problem

$$\min_u \int_\Omega |\nabla u|^2 \, dx$$

subject to constraints

$$\int_{\partial\Omega_1} u \, ds - \int_{\partial\Omega_2} u \, ds = V, \tag{40}$$

We do not pose constraints to account for the insulation condition on the horizontal parts $\partial\Omega_3$ and $\partial\Omega_4$ because it is a natural condition that will be automatically satisfied at the optimal solution.

Accounting for the above constraints (40) with Lagrange multipliers λ we end up with the variational problem

$$I = \min_u \left[\int_\Omega |\nabla u|^2 \, dx + \lambda \left(\int_{\partial\Omega_1} u \, ds - \int_{\partial\Omega_2} u \, ds - V \right) \right].$$

The calculation of the first variation of I with respect to u leads to the Euler equation in the domain

$$\Delta u = 0 \quad \text{in } \Omega$$

and the boundary conditions

$$\frac{\partial u}{\partial n} + \lambda = 0 \quad \text{on } \partial\Omega_1, \quad \frac{\partial u}{\partial n} - \lambda = 0 \quad \text{on } \partial\Omega_2,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_3, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_4$$

One can see that the boundary currents (the normal derivatives of the potential) are constant for the optimal solution. This is the missing boundary condition.

Finally, we solve the boundary value problem assuming that λ is know. The potential is an affine function $u = C + \lambda x_1$, where C is a constant. The potential difference is $V = \lambda a$, which gives $\lambda = \frac{V}{a}$.

Problem 3.1 Consider the previous problem assuming that the domain Ω is a sector of the annulus $r_0 \leq r_1$, $\theta_0 \leq \theta \leq \theta_1$ that conducts the fixed total current from one radial cut to another.

Problem 3.2 Consider the previous problem assuming that the conductivity is inhomogeneous. Derive the boundary conditions.

4 General form of a variational functional

4.1 Reduction to isoperimetric problem

Lagrange method allows for reformulation of an extremal problem in a general form as a simplest variational problem. The minimizing functional can be the product, ratio, superposition of other differentiable function of integrals of the minimizer and its derivative. Consider the problem

$$J = \min_u \Phi(I_1, \dots, I_n) \tag{41}$$

where

$$I_k(u) = \int_a^b F_k(x, u, u')dx \quad k = 1, \dots n \tag{42}$$

and Φ is a continuously differentiable function. Using Lagrange multipliers λ_1, λ_n, we transform the problem (41) to the form

$$J = \min_u \min_{I_1,\dots,I_n} \max_{\lambda_1,\dots\lambda_n} \left\{ \Phi + \sum_{k=1}^n \lambda_k \left(I_k - \int_a^b F_k(x, u, u')dx \right) \right\}. \tag{43}$$

The stationarity conditions for (43) consist of n algebraic equations

$$\frac{\partial \Phi}{\partial I_k} + \lambda_i = 0 \tag{44}$$

and the differential equation – the Euler equation

$$S(\Psi, u) = 0$$

$$\left(\text{recall that } S(\Psi, u) = \frac{d}{dx}\frac{\partial \Psi}{\partial u'} - \frac{\partial \Psi}{\partial u} \right)$$

for the function

$$\Psi(u) = \sum_{k=1}^n \lambda_k F_k(x, u, u')$$

Together with the definitions (42) of I_k, this system enables us to determine the real parameters I_k and λ_k and the function $u(x)$. The Lagrange multipliers can be excluded from the previous expression using (44), then the remaining stationary condition becomes an integro-differential equation

$$S(\bar{\Psi}, u) = 0, \quad \bar{\Psi}(I_k, u) = \sum_{k=1}^n \frac{\partial \Phi}{\partial I_k} F_k(x, u, u') \tag{45}$$

Next examples illustrate the approach.

4.1.1 The product of integrals

Consider the problem

$$\min_u J(u), \quad J(u) = \left(\int_a^b \phi(x, u, u')dx \right) \left(\int_a^b \psi(x, u, u')dx \right).$$

We rewrite the minimizing quantity as

$$J(u) = I_1(u) I_2(u), \quad I_1(u) = \int_a^b \phi(x, u, u')dx, \quad I_2(u) = \int_a^b \psi(x, u, u')dx,$$

apply stationary condition (45), and obtain the condition

$$I_1 \delta I_2 + I_2 \delta I_1 = I_2(u) S(\phi(u), u) + I_1(u) S(\psi(u), u) = 0. \tag{46}$$

or

$$\left(\int_a^b \phi(x, u, u')dx \right)^{-1} S(\phi(u), u) + \left(\int_a^b \psi(x, u, u')dx \right)^{-1} S(\psi(u), u) = 0$$

The equation is nonlocal: Solution u at each point depends on its first and second derivatives and integrals of $\phi(x, u, u')$ and $\phi(x, u, u')$ over the whole interval $[a, b]$.

Example 4.1 Solve the problem

$$\min_u \left(\int_0^1 (u')^2 dx \right) \left(\int_0^1 (u + 1) dx \right) \quad u(0) = 0, \quad u(1) = a$$

We denote

$$I_1 = \int_0^1 (u')^2 dx, \quad I_2 = \int_0^1 (u + 1) dx$$

and compute the Euler equation using (46)

$$I_2 u'' - I_1 = 0, \quad u(0) = 0, \quad u(1) = a.$$

or

$$u'' - R = 0, \quad u(0) = 0, \quad u(1) = a, \quad R = \frac{I_1}{I_2}$$

The integration gives

$$u(x) = \frac{1}{2} R x^2 + \left(a - \frac{1}{2} R \right) x,$$

We obtain the solution that depends on R – the ratio of the integrals of two function of this solution. To find R, we substitute the expression for $u = u(R)$ into right-hand sides of I_1 and I_2,

$$I_1 = \frac{R^2}{12} + a^2, \quad I_2 = -\frac{R}{12} + \frac{1}{2} a + 1$$

compute the ratio, $\frac{I_1}{I_2} = R$ and obtain the equation for R,

$$R = \frac{R^2 + 12a^2}{R + 6a + 12}$$

Solving it, we find $R = \frac{1}{2}(3a + 6 \pm \sqrt{36 + 36a - 15a^2})$.

At this point, we do not know whether the solution correspond to minimum or maximum. This question is investigated in Chapter **??**.

4.1.2 The ratio of integrals

Consider the problem

$$\min_u J(u), \quad J(u) = \frac{\int_a^b \phi(x,u,u')dx}{\int_a^b \psi(x,u,u')dx}.$$

We rewrite it as

$$J = \frac{I_1}{I_2}, \quad I_1(u) = \int_a^b \phi(x,u,u')dx, \quad I_2(u) = \int_a^b \psi(x,u,u')dx, \tag{47}$$

apply stationary condition (45), and obtain the condition

$$\frac{1}{I_2(u)} S(\phi(u),u) - \frac{I_1(u)}{I_2^2(u)} S(\psi(u),u) = 0.$$

Multiplying this equality by I_2 and using definition (47) of the goal functional, we bring the previous expression to the form

$$S(\phi,u) - J\,S(\psi,u) = S(\phi - J\psi, u) = 0$$

Observe that the stationarity condition depends on the cost J of the problem.

Example 4.2 Solve the problem

$$\min_u J(u), \quad J = \frac{\int_0^1 (u')^2 dx}{\int_0^1 (u-1)dx} \quad u(0)=0, \quad u(1)=a$$

We compute the Euler equation

$$u'' - J = 0, \quad u(0) = 0, \quad u(1) = a.$$

where

$$R = \frac{I_1}{I_2}, \quad I_1 = \int_0^1 (u')^2 dx, \quad I_2 = \int_0^1 (u-1)dx$$

The integration gives

$$u(x) = \frac{1}{2}Rx^2 + \left(a - \frac{1}{2}R\right)x,$$

We obtain the solution that depends on R – the ratio of the integrals of two function of this solution. To find R, we substitute the expression for $u = u(R)$ into right-hand sides of I_1 and I_2,

$$I_1 = \frac{R^2}{12} + a^2, \quad I_2 = -\frac{R}{12} + \frac{1}{2}a + 1$$

compute the ratio, $\frac{I_1}{I_2} = R$ and obtain the equation for R,

$$R = \frac{R^2 + 12a^2}{R + 6a + 12}$$

Solving it, we find $R = \frac{1}{2}(3a + 6 \pm \sqrt{36 + 36a - 15a^2})$.

At this point, we do not state whether the solution correspond to minimum or maximum. This question is investigated in Chapter ??.

4.1.3 Superposition of integrals

Consider the problem

$$\min_u \int_a^b R\left(x, u, u', \int_a^b \phi(x, u, u')dx\right) dx$$

We introduce a new variable I

$$I = \int_a^b \phi(x, u, u')dx$$

and reformulate the problem as

$$\min_u \int_a^b \left[R(x, u, u', I) + \lambda\left(\phi(x, u, u') - \frac{I}{b-a}\right)\right] dx$$

where λ is the Lagrange multiplier. The stationarity conditions are:

$$S((R + \lambda\phi), u) = 0, \quad \frac{\partial R}{\partial I} - \frac{1}{b-a} = 0.$$

and the above definition of I.

Example 4.3 (Integral term in the Lagrangian) Consider the following extremal problem posed in "physical terms": Find the function $u(x)$ at the interval $[0, 1]$ that is has prescribed values at its ends,

$$u(0) = 1, \quad u(1) = 0, \tag{48}$$

has a smallest L_2-norm

$$\int_0^1 u'^2 dx$$

of the derivative u', and stays maximally close to its averaged over the interval $[0, x]$ value a,

$$a = \int_0^x u(t)dt \tag{49}$$

In order to formulate a mathematical extremal problem, we combine the two above requests on $u(x)$ the into one Lagrangian F equal to the weighted sum of them:

$$F = u'^2 + \alpha\left(u - \int_0^1 u(t)dt\right)^2, \quad u(0) = 1, \quad u(1) = 0$$

where $\alpha \geq 0$ is a weight coefficient that show the relative importance of the two criteria. Function $u(x)$ is a solution to the extremal problem

$$\min_{u(x),\ u(0)=1, u(1)=0} \int_0^1 F\left(u, u', \int_0^1 u(t)dt\right) dx \tag{50}$$

We end up with the variational problem with the Lagrangian that depends on the minimizer u, its derivative and its integral.

Remark 4.1 Formulating the problem, we could include boundary conditions into a minimized part of the functional instead of postulating them; in this case the problem would be

$$\min_{u(x)} \left\{ \int_0^1 F\left(u, u', \int_0^1 u(t)dt\right) dx + \beta_1 (u(0)-1)^2 + \beta_2 u(1)^2 \right\}$$

where $\beta_1 \geq 0$ and $\beta_2 \geq 0$ are the additional weight coefficients.

We bring the problem (50) to the form of the standard variational problem, accounting for the equality (49) with the Lagrange multiplier λ; the objective functional becomes

$$J = \int_0^1 \left(u'^2 + \alpha(u-a)^2\right) dx + \lambda \left(a - \int_0^1 u\, dx\right)$$

or

$$J = \int_0^1 \left(u'^2 + \alpha(u-a)^2 + \lambda(a-u)\right) dx$$

The parameter a and the function $u(x)$ are the unknowns. The stationary condition with respect to a is

$$\frac{\partial J}{\partial a} = \int_0^1 (-2\alpha(u-a) + \lambda)\, dx = 2\alpha a + \lambda - 2 \underbrace{\int_0^1 u\, dx}_{=a} = 0,$$

it allows for linking a and λ,

$$\lambda = 2(\alpha - 1)a.$$

The stationary condition with respect to $u(x)$ (Euler equation) is

$$2u'' - 2\alpha(u-a) - \lambda = 0$$

We exclude λ using the obtained expression for λ, and obtain

$$2u'' - 2\alpha u + a = 0 \tag{51}$$

The integro-differential system (49) and (51) with the boundary conditions (48) determines the minimizer.

To solve the system, we first solve (51) and (48) treating a as a parameter,

$$u(x) = \frac{a}{2\alpha} + A \sinh(\sqrt{\alpha}x) + B \cosh(\sqrt{\alpha}x)$$

where

$$A = \left(\frac{a}{2\alpha} - 1\right) \frac{\cosh(\sqrt{\alpha})}{\sinh(\sqrt{\alpha})}, \quad B = 1 - \frac{a}{2\alpha},$$

and substitute this solution into (49) obtaining the linear equation for the remaining unknown a. We have

$$u(x) = c_1(x)a + c_2(x)$$

where

$$c_1(x) = \frac{1}{2\alpha}\left(1 + \frac{\cosh(\sqrt{\alpha})}{\sinh(\sqrt{\alpha})}\sinh(\sqrt{\alpha}x) - \cosh(\sqrt{\alpha}x)\right)$$

and

$$c_2(x) = \left(\cosh(\sqrt{\alpha}x) - \frac{\cosh(\sqrt{\alpha})}{\sinh(\sqrt{\alpha})}\sinh(\sqrt{\alpha}x)\right)$$

and (49) becomes

$$a = a\int_0^1 c_1(x)dx + \int_0^1 c_2(x)dx$$

which implies

$$a = \frac{\int_0^1 c_2(x)dx}{\int_0^1 c_1(x)dx - 1}$$

The general procedure is similar: We always can rewrite a minimization problem in the standard form adding new variables (as the parameter c in the previous examples) and corresponding Lagrange multipliers.

Inequality in the isoperimetric condition Often, the isoperimetric constraint is given in the form of an inequality

$$\min_u \int_a^b F(x,u,u')dx \quad \text{subject to} \quad \int_a^b G(x,u,u')dx \geq 0 \tag{52}$$

In this case, the additional condition $\lambda \geq 0$ is added to the Euler-Lagrange equations (**??**) according to the (**??**).

Remark 4.2 Sometimes, the replacement of an equality constraint with the corresponding inequality can help to determine the sign of the Lagrange multiplier. For example, consider the Dido problem, and replace the condition that the perimeter is fixed with the condition that the perimeter is smaller than or equal to a constant. Obviously, the maximal area corresponds to the maximal allowed perimeter and the constraint is always active. On the other hand, the problem with the inequality constraint requires positivity of the Lagrange multiplier; so we conclude that the multiplier is positive in both the modified and original problem.

4.2 Homogeneous functionals and Eigenvalue Problem

The next two problems are *homogeneous*: The functionals do not vary if the solution is multiplied by any number. Therefore, the solution is defined up to a constant multiplier.

The eigenvalue problem corresponds to the functional

$$I_1 = \min_u \frac{\int_0^1 (u')^2 dx}{\int_0^1 u^2 dx} \quad x(0) = x(1) = 0 \tag{53}$$

it can be compared with the problem:

$$I_2 = \min_u \frac{\int_0^1 (u')^2 dx}{\left(\int_0^1 u dx\right)^2} \quad x(0) = x(1) = 0 \tag{54}$$

Do these problem have nonzero solutions?

Consider the problem (53). Because the solution is defined up to a multiplier, we can normalize it assuming that

$$\int_0^1 u^2 dx = 1 \tag{55}$$

Then the problem takes the form

$$I_1 = \min_u \int_0^1 \left((u')^2 + \lambda u^2\right) dx \quad x(0) = x(1) = 0$$

where λ is the Lagrange multiplier by the normalization constraint (55). The Euler equation is

$$u'' - \lambda u = 0, \quad x(0) = x(1) = 0$$

This equation represents the eigenvalue problem. It has nonzero solutions u only if λ takes special values – the eigenvalues. These values are $\lambda_n = -(\pi n)^2$ where n is a nonzero integer; the corresponding solutions – the eigenfunctions u_n – are equal to $u_n(x) = C \sin(\pi n x)$. The constant C is determined from the normalization (55) as $C = \sqrt{2}$. The cost of the problem at a stationary solution u_n is

$$\int_0^1 (u_n')^2 dx = n^2\pi^2$$

The minimal cost I_1 corresponds to $n = 1$ and is equal to $I_1 = \pi^2$

The problem (54) is also homogeneous, and its solution u is defined up a multiplier. We reformulate the problem by normalizing the solution,

$$\int_0^1 u dx = 1.$$

The problem (54) becomes

$$\min_u \int_0^1 \left((u')^2 + \lambda u\right) dx \quad x(0) = x(1) = 0$$

where λ is the Lagrange multiplier by the normalization constraint.

The minimizer u satisfies the Euler equation

$$u'' - \frac{\lambda}{2} = 0, \quad x(0) = x(1) = 0$$

and is equal to $u = \frac{\lambda}{2}x(x-1)$. The constraint gives $\lambda = 12$ and the objective is

$$\int_0^1 (u')^2 dx = \int_0^1 (6-12x)^2 dx = 12$$

These two homogeneous variational problems correspond to different types of Euler equation. The equation for the problem (53) is homogeneous; it has either infinitely many solutions or no solutions depending on λ. It can select the stationary solution set but cannot select a solution inside the set: this is done by straight comparison of the objective functionals. The problem (54) leads to hon-homogeneous Euler equation that linearly depend on the constant λ of normalization. It has a unique solution if the normalization constant is fixed.

Homogeneous with a power functionals To complete the considerations, consider a larger class of homogeneous with a power p functionals, $I(qu) = q^p I(u)$ where $q > 0$ is an arbitrary positive number. For example function $I(x) = ax^4$ is homogeneous with the power four, because $I(qx) = aq^4x^4 = q^4 I(x)$. Here, $p \neq 1$ is a real number. for all u. For example, the functional can be equal to

$$J_3(u) = \frac{\int_0^1 (u')^2 dx}{\left|\int_0^1 u dx\right|^p}, \quad x(0) = x(1) = 0, \quad u \not\equiv 0 \tag{56}$$

which implies that it is homogeneous with the power $2-p$, because $J_3(qu) = q^{2-p} J_3(u)$.

The minimization of such functionals leads to a trivial result: Either $\inf_u J_3 = 0$ or $\inf_u J_3 = -\infty$, because the positive factor q^p can be make arbitrarily large or small.

More exactly, if there exist u_0 such that $I(u_0) \leq 0$, than $\inf_u J_3 = -\infty$; the minimizing sequence consists of the terms $q_k u_0$ where the multipliers q_k are chosen so that $\lim q_k^p = \infty$.

If $I(u_0) \geq 0$ for all u_0, than $\inf_u J_3 = 0$; the minimizing sequence again consists of the terms $q_k u_0$ where the multipliers q_k are chosen so that $\lim q_k^p = 0$.

Remark 4.3 In the both cases, the minimizer itself does not exist but the minimizing sequence can be built. These problems are examples of variational problems without classical solution that satisfies Euler equation. Formally, the solution of problem (56) does not exist because the class of minimizers is open: It does not include $u \equiv 0$ and $u \equiv \infty$ one of which is the minimizer. We investigate the problems without classical solutions in Chapter **??**.

4.3 Constraints in boundary conditions

Constraints on the boundary, fixed interval Consider a variational problem (in standard notations) for a vector minimizer $\boldsymbol{u}$. If there are no constrains imposed on the end of the trajectory, the solution to the problem satisfies n natural boundary conditions

$$\delta \boldsymbol{u}(b) \cdot \left. \frac{\partial F}{\partial \boldsymbol{u}'} \right|_{x=b} = 0$$

(For definiteness, we consider here conditions on the right end, the others are clearly identical).

The vector minimizer of a variational problem may have some additional constraints posed at the end point of the optimal trajectory. Denote the boundary value of $u_i(b)$ by v_i The constraints are

$$\phi_i(v_1, \ldots v_n) = 0 \quad i = 1, \ldots, k; \quad k \le n$$

or in vector form,

$$\Phi(x, \boldsymbol{v}) = 0,$$

where Φ is the corresponding vector function. The minimizer satisfies these conditions and $n - k$ supplementary natural conditions that arrive from the minimization requirement. Here we derive these supplementary boundary conditions for the minimizer.

Let us add the constrains with a vector Lagrange multiplier $\boldsymbol{\lambda} = (\lambda_1, \ldots . \lambda_k)$ to the problem. The variation of $\boldsymbol{v} = \boldsymbol{u}(b)$ gives the conditions

$$\delta \boldsymbol{v} \cdot \left[\left. \frac{\partial F}{\partial \boldsymbol{u}'} \right|_{x=b, \boldsymbol{u}=\boldsymbol{v}} + \frac{\partial \Phi}{\partial \boldsymbol{v}} \boldsymbol{\lambda} \right] = 0$$

The vector in the square brackets must be zero because of arbitrariness of $\nu = \delta \boldsymbol{u}(b)$.

Next, we may exclude $\boldsymbol{\lambda}$ from the last equation (see the previous section 2.2):

$$\boldsymbol{\lambda} = - \left[\left(\frac{\partial \Phi}{\partial \boldsymbol{u}} \right)^T \frac{\partial \Phi}{\partial \boldsymbol{u}} \right]^{-1} \left. \frac{\partial F}{\partial \boldsymbol{u}'} \right|_{x=b, \boldsymbol{u}=\boldsymbol{v}} \tag{57}$$

and obtain the conditions

$$\left(I - \frac{\partial \Phi}{\partial \boldsymbol{u}}^T \left[\left(\frac{\partial \Phi}{\partial \boldsymbol{u}} \right)^T \frac{\partial \Phi}{\partial \boldsymbol{u}} \right]^{-1} \frac{\partial \Phi}{\partial \boldsymbol{u}} \right) \left. \frac{\partial F}{\partial \boldsymbol{u}'} \right|_{x=b, \boldsymbol{u}=\boldsymbol{v}} = 0 \tag{58}$$

The rank of the matrix in the parenthesis is equal to $n - k$. Together with k constrains, these conditions are the natural conditions for the variational problem.

4.3.1 Example

$$\min_{u_1, u_2} \int_a^b (u_1'^2 + u_2'^2 + u_3') dx, \quad u_1(b) + u_2(b) = 1, \; u_1(b) - u_3(b) = 1,$$

We compute

$$\frac{\partial F}{\partial \boldsymbol{u}'} = \begin{pmatrix} 2u_1 \\ 2u_2 \\ 1 \end{pmatrix}, \quad \frac{\partial \Phi}{\partial \boldsymbol{u}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(please continue..)

Free boundary with constraints Consider a general case when the constraints $\Phi(x, u) = 0$ are posed on the solution at the end point. Variation of these constrains results in the condition:

$$\delta\Phi(x, u)|_{x=b} = \frac{\partial \Phi}{\partial u}\delta u + \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u}u'\right)\delta x$$

Adding the constraints to the problem with Lagrange multiplier $\boldsymbol{\lambda}$, performing variation, and collecting terms proportional to δx, we obtain the condition at the unknown end point $x = b$

$$F(x, u, u') - \frac{\partial F}{\partial u'}u' + \boldsymbol{\lambda}^T \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u}u'\right) = 0$$

where $\boldsymbol{\lambda}$ is defined in (57). Together with $n-k$ conditions (58) and k constraints, they provide $n + 1$ equations for the unknowns $u_1(b), \ldots, u_n(b), b$.